Limity

Substituce za polární souřadnice

$$x = x_0 + r\cos\varphi$$
$$y = y_0 + r\sin\varphi$$
$$(x_0, y_0) = (0,0)$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{\sqrt{x^2+y^2}}$$

$$r^2(1)$$

$$\lim_{r\to 0} \frac{r^2\cos^2\varphi - r^2\sin^2\varphi}{r} = \lim_{r\to 0} r\cos^2\varphi - r\sin\varphi = 0$$

Počítáme se po kružnicích se zmenšujícím se r.

$$\lim_{(x,y)\to(0,0)} \frac{x\sin y}{x^2+y^2} =$$

$y = kx \quad x = 0$

$x = y$

$$\lim \frac{x\sin x}{x^2+x^2} \lim \frac{x\sin x}{2xx} = \lim \frac{\sin x}{2x} = \frac{1}{2}$$

$x = 0$

$$\lim \frac{0}{y^2} = 0$$

limita neexistuje

(e) $$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^3+y^3}$$

$x = -y \quad f(x,y) = \frac{A(a,b)}{g(x,y)}$

M: $x = -y \wedge (x,y) = (0,0)$

$\forall a \in M \; g(a) = 0 \wedge h(a) \neq 0$

$a_0 = (0,0)$ je hromadný vod. ozn.

$y = kx$

$$\frac{x^4 + k^4x^4}{x^3 + k^3x^3} = \frac{x^4(1+k^4)}{x^3(1+k^3)} = \frac{x(1+k^4)}{(1+k^3)} \xrightarrow{x\to 0} 0$$

$x = 0$

$$\lim_{y\to 0} \frac{y^4}{y^3} = \lim_{y\to 0} y = 0$$

$$\frac{tg(x\cdot 0)}{y} \xrightarrow{(4,0)} = \frac{tg(xy)\cdot x}{xy} \qquad 1\cdot x = 4$$

$$\frac{tg(z)}{(z)} = 1$$

Taylorův polynom druhého řádu

$$f(x,y) = e^{x^2+2y^2} - 2\cos(x+y) \qquad \text{bod } a_0 = (0,0)$$

$$\nabla f(x,y) = \left(2xe^{x^2+2y^2} + 2\sin(x+y),\ 4ye^{x^2+2y^2} + 2\sin(x+y)\right)$$

$$H_f = \begin{pmatrix} 2e^{x^2+2y^2} + 4x^2e^{x^2+2y^2} + 2\cos(x+y) & 4xy \\ \cdots & \cdots \end{pmatrix}$$

$$\vec{h} = (a-a_0) = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$T(a_0+\vec{h}) = \underbrace{f(a_0)}_{-1} + \underbrace{f'(a_0)\cdot\vec{h}}_{0} + \underbrace{\vec{h}\,\frac{1}{2}H_f(a_0)\,\vec{h}}_{>0}$$

$$\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}$$

nadhlenovým vodem matice M

$$\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \quad \Delta_1 = 4 \quad \Delta_2 = 24-4 = 20 > 0$$

minimum

(2.) $M: \frac{x^2}{4} + \frac{y^2}{9} \le 1 \quad g(x)$

hledáme vazbu

$z = 2x - y$ nejvyšší výšky.

$f(x)$

$$\partial M: \frac{x^2}{4} + \frac{y^2}{9} - 1 = g(x)$$

$$z = f(x,y) = 2x - y$$

$$L(x,y) = 2x - y - z - \lambda\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right)$$

$$\frac{\partial f}{\partial x} = 2 \qquad \frac{\partial f}{\partial y} = -1$$

$$\frac{\partial g}{\partial x} = \frac{x}{2} \qquad \frac{\partial g}{\partial y} = \frac{2}{9}y$$

$$2 = \lambda\frac{x}{2}$$

$$-1 = \lambda\frac{2}{9}y$$

$$-2 = \frac{9x}{4y}$$

$$\lambda = \frac{4}{x} \qquad x \neq 0 \quad y = \pm 3$$

$$\lambda = -\frac{9}{2y} \qquad y \neq 0 \quad y = \pm 2$$

$$\frac{4}{x} = -\frac{9}{2y}$$

$$8y = -9x$$

$$x = -\frac{8}{9}y$$

$$y = \frac{-9x}{8}$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$\frac{x^2}{4} + \frac{9x^2}{8^2} = 1$$

$$\frac{x^2}{4} + \frac{9x^2}{64} = 1$$

$$\frac{25x^2}{64} = 1$$

$$25x^2 = 64$$

$$5x = \pm 8$$

$$x = \pm\frac{8}{5}$$

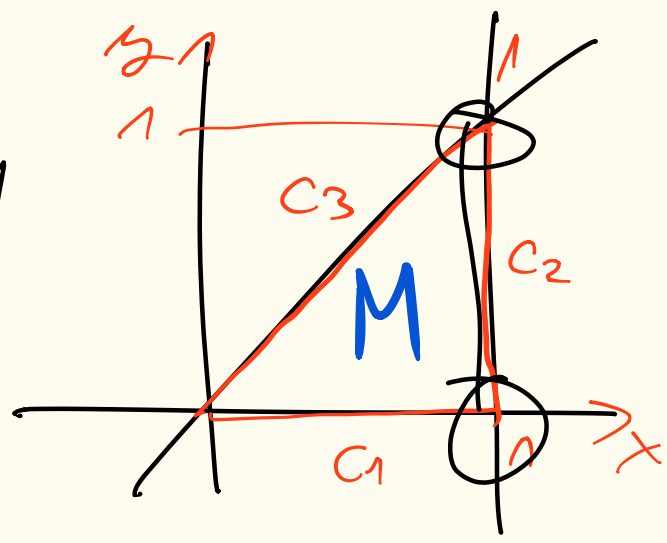
$$y = \frac{-9\cdot\frac{8}{5}}{8} = \mp\frac{9}{5}$$

Dva extrémy $\left(\frac{8}{5}, -\frac{9}{5}\right)$

$\left(-\frac{8}{5}, \frac{9}{5}\right)$

$f(x,y) = x^2 - xy + x + y$ $\begin{cases} (0,0) = 0 \\ (1,1) = 2 \\ (1,0) = 2 \end{cases}$

$M: 0 \le y \le x \;\&\; x \le 1$



$M^0: 0 < y < x \quad x < 1$

$\nabla f(x,0) = (2x - y + 1, -x + 1) = (0,0)$

$\varphi_1(t) = (t,0)\ t \in (0,1)$

$\varphi_2(t) = (1,t)\ t \in (0,1)$

$\varphi_3(t) = (t,t)\ t \in (0,1)$

$2x - y + 1 = 0 \qquad 2 - y + 1 = 0$

$y = 3$

$-x + 1 = 0 \qquad x = 1$

X

Neex. extrém na M^0

$\partial M: \quad f_1(\varphi_1(t)) = t^2 + t \qquad f_1'(t) = 2t + 1 = 0$

$2t = -1$

$t = -1/2$ ∅

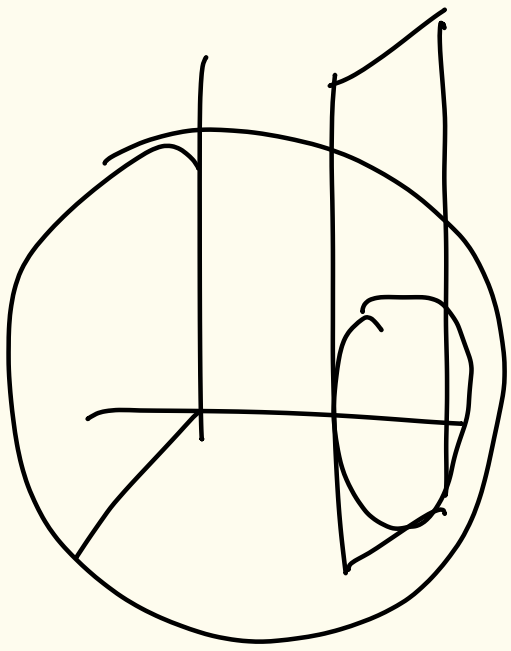
$f_2(\varphi_2(t)) = 1 - t + 1 + t = 2 \qquad f_2'(t) = 0$ všude

$f_3(\varphi_3(t)) = t^2 - t^2 + 2t = 2t \qquad f_3'(t) = 2$ nikde ∅

Hmotnost dutiny ve tvaru kuželce

$x^2 + y^2 + z^2 = 1 \qquad x = 1/2$

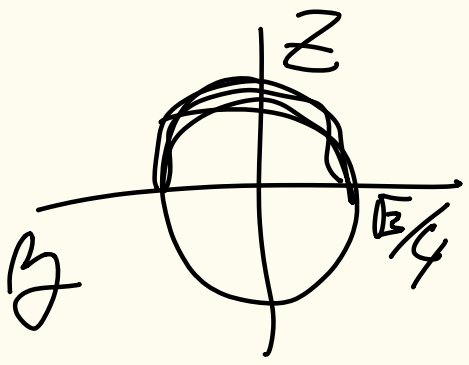
délková hustota $\delta(x,y,z) = y^2 + |z|$



$y^2 + z^2 = 3/4$

$y = \sqrt{3}/2 \cos\varphi$

$\varphi(t)\ (1/2, y, z) \qquad z = \sqrt{3}/2 \sin\varphi$



$4 \int_0^{\pi/2} \varphi(t) \cdot \|\varphi'(t)\| \, dt = 4($

2020/2/1

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

$x = 0$

$$\lim_{y\to 0} \frac{0}{y} = 0$$

$x = r\cos\varphi$

$y = r\sin\varphi$

$$\lim_{r\to 0} \frac{r^2\cos\varphi\sin\varphi}{\sqrt{r^2}} = \lim_{r\to 0} \frac{r^2\cos\varphi\sin\varphi}{r} = \lim_{r\to 0} r\cos\varphi\sin\varphi = 0$$

lze odhadnout, a je ono $\frac{0}{0}$

$y = x$

$$\lim_{x\to 0} \frac{x^2}{\sqrt{2x^2}} = \lim_{x\to 0} \frac{x}{\sqrt{2}} = 0$$

$y = kx$

$$\lim_{x\to 0} \frac{xkx}{\sqrt{x^2+k^2x^2}} = \frac{kx^2}{x\sqrt{k^2+1}} \xrightarrow{x\to 0} 0$$

$x^2(k^2+1)$

$\frac{h(x)}{g(x)}$

$$\lim_{(x,y)\to(0,0)} \frac{\tan(x)}{y} \quad \frac{\sin x}{\cos x}$$

$x = 0 \quad \frac{0}{y} \to 0$

$y = \sin x$

$\frac{\sin x}{\cos x\sin x} = \frac{1}{\cos x} \to 1$

nemá limitu

(4) $\vec{r}(t) = (e^{t+1}, e^t\sin t, e^t\cos t) \quad 0 \le t \le 2$

$$\int_C 1\,ds = \int_0^2 1 \cdot \underbrace{\|\varphi'(t)\|}_{(e^{t+1}, e^t\sin t + e^t\cos t, e^t\cos t - e^t\sin t)} dt$$

$$\|\varphi'(t)\| = e^{2t+2} + e^{2t}\sin^2 t + 2e^{2t}\sin t\cos t + e^{2t}\cos^2 t + e^{2t}\cos^2 t$$

$$= e^{2t+2} + e^{2t} + e^{2t}$$

$$= \sqrt{e^{2t}(e^2+2)} = e^t \sqrt{e^2+2}$$

$-2e^{2t}\sin t\cos t + e^{2t}\sin^2 t$

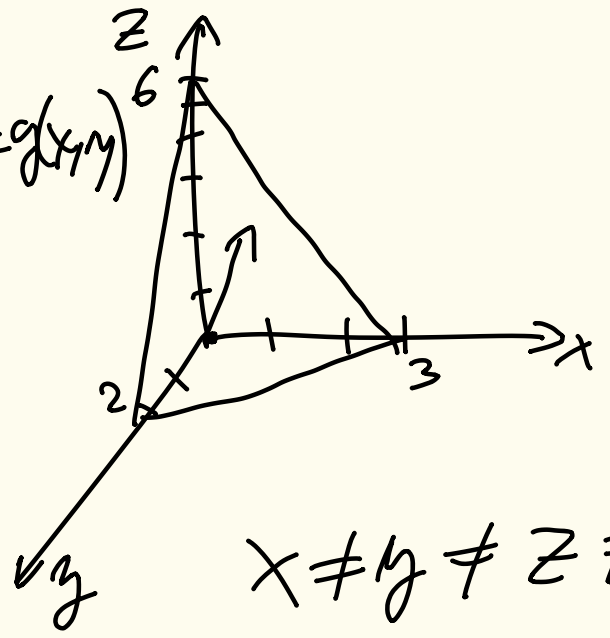
$$\int_0^2 e^t \sqrt{e^2+2}\, dt = \sqrt{e^2+2}\,[e^t]_0^2 = \sqrt{e^2+2}\,(e^2-1)$$

(2) Do ćtyřstěnu:

$x=0$
$y=0$
$z=0$

$2x+3y+z=6$

$g(x)$ $\quad 6-2x-3y = z = g(x,y)$



$z = 6-2x-3y$

$f(x,y) = xy(6-2x-3y)$

$6xy - 2x^2y - 3xy^2$

$\nabla f(6y-4xy-3y^2,\ 6x-2x^2-6xy) = (0,0)$

$6y-4xy-3y^2=0 \qquad 4xy = 6y-3y^2$

$6x-2x^2-6xy=0 \qquad xy = \frac{3}{2}y - \frac{3}{4}y^2$

$y(6-4x-3y)=0$

$y=0 \ \| \ 6-4x-3y=0$

$x(6-2x-6y)$

$x=0 \ \| \ 6-2x-6y$

$x \neq y \neq z \neq 0$

$$6-4x-3y=0$$

$$6-2x-6y=0$$

$$-6+9y$$

$$9y=6$$

$$y=\frac{6}{9}=\frac{2}{3}$$

$$6-4x-2=0$$

$$4x=4$$

$$x=1$$

$$z=6-2x-3y=6-2-2=2$$

$$(x,y,z)=(1,\tfrac{2}{3},2)$$

$$V_{\square}=4/3$$

(3)

$$\int_0^1 \int_1^{2-x} f\,dy\,dx$$

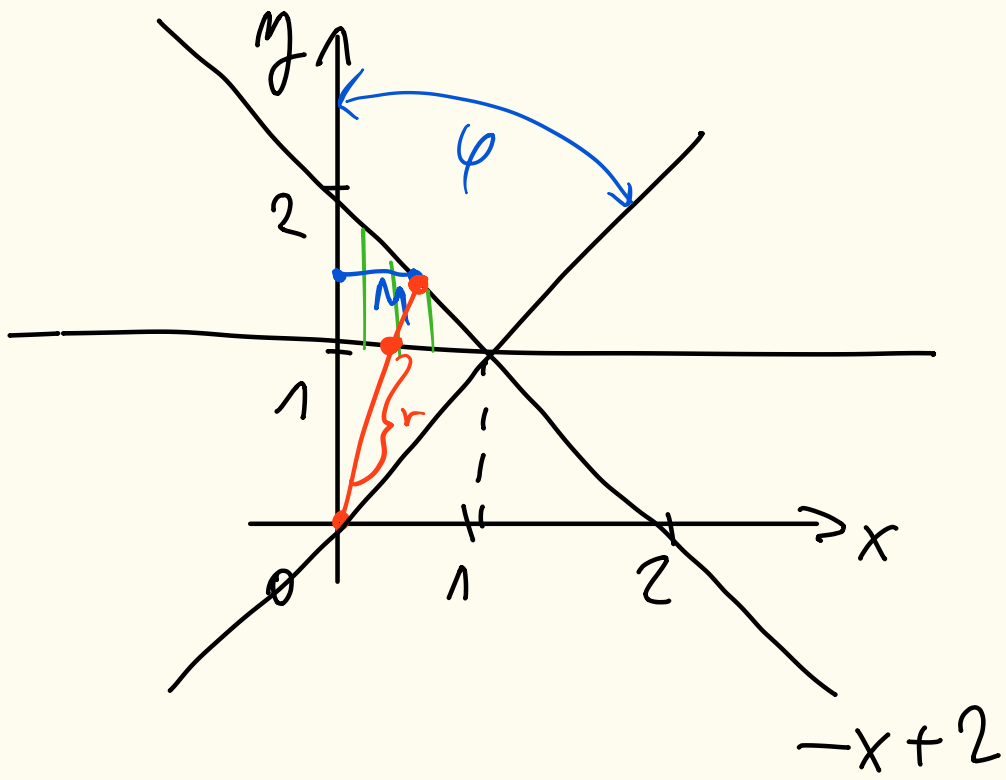
U: $0\le x\le 2-y$

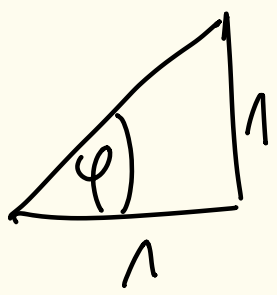
$1\le y\le 2$

$$\int_1^2 \int_0^{2-y} f(x,y)\,dx\,dy$$

U: $\frac{\pi}{4}\le\varphi\le\frac{\pi}{2}$

$\frac{1}{\sin\varphi}\le r\le\frac{2}{\sin\varphi+\cos\varphi}$





$\tan 1$

$\operatorname{arctan} 1=\frac{\pi}{4}$

$x=r\cos\varphi$

$y=r\sin\varphi$

$y=-x+2$

$x=2-y$

$y=1$

$r\sin\varphi=1$

$r=\frac{1}{\sin\varphi}$

$$\int_{\frac{\pi}{4}}^{\pi/2}\int_{1/\sin\varphi}^{2/\sin\varphi+\cos\varphi} f(r\cos\varphi, r\sin\varphi)\cdot r\,dr\,d\varphi$$

$$y=-x+2$$

$$r\sin\varphi=-r\cos\varphi+2$$

$$r\sin\varphi+r\cos\varphi=2$$

$$r=\frac{2}{\sin\varphi+\cos\varphi}$$

Test 2020 - 3

(1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x\cos y}{\sqrt{x^2+y^2}}$

$x=0$ $\lim \dfrac{0}{y}=0$

$y=kx$

$$\frac{x\cos kx}{\sqrt{x^2+k^2x^2}}=\frac{x\cos kx}{x\sqrt{1+k^2}}=$$

$$=\frac{\cos kx}{\sqrt{1+k^2}} \xrightarrow[x\to 0]{} \frac{1}{\sqrt{1+k^2}}$$

$\Rightarrow$ nema limita

$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^4}{x^4+y^4}$

$x=0$

$$\frac{0}{y^4}=0$$

$y=kx$

$$\frac{xk^4x^4}{x^4+k^4x^4}=\frac{xk^4x^4}{x^4(1+k^4)}=\frac{xk^4}{1+k^4}\xrightarrow{x\to 0}0$$

$\lim\limits_{r\to 0} \dfrac{r\cos\varphi\, r^4\sin^4\varphi}{r^4\cos^4\varphi+r^4\sin^4\varphi}=\dfrac{r^5\cos\varphi\sin^4\varphi}{\cos^4\varphi+\sin^4\varphi}\xrightarrow{r\to 0}0$

ovdje ograničena

protra i $\lim\limits_{(x,y)\to(0,0)}\dfrac{xy^4}{x^4+y^4}=0$

(2)

$f(x,y) = x^2 + xy^2 - 16x$

nutná podmínka existence extrému

$\nabla f(x,y) = (2x + y^2 - 16, 2xy) = (0,0)$

$2x + y^2 - 16 = 0$

$2xy = 0$

$2x = 16 - y^2$

$(16 - y^2)y = 0$

$y = 0$ $\quad 2x = 16$

$x = 8$

$y = \pm 4$

$2x = 16 - 16 = 0$

$Hf(x,y) = \begin{pmatrix} 2 & 2y \\ 2y & 2x \end{pmatrix}$

$Hf(0,8) = \begin{pmatrix} 2 & 0 \\ 0 & 16 \end{pmatrix}$ $\quad (8,0)$

$\Delta_1 = 2 > 0$

$\Delta_2 = 32 > 0$ $\quad (0,-4)$

pos. def. $\quad (0,4)$

(kvadratická forma)

$f(8,0) = 64 + 0 - 16 \cdot 8$

$= -64 - 80$

$- 48$

$- 128$

$Hf(4,0) = \begin{pmatrix} 2 & -8 \\ -8 & 0 \end{pmatrix}$

$\Delta_1 = 2 > 0$

$\Delta_2 = -64 > 0$

indefinitní

sedlový bod

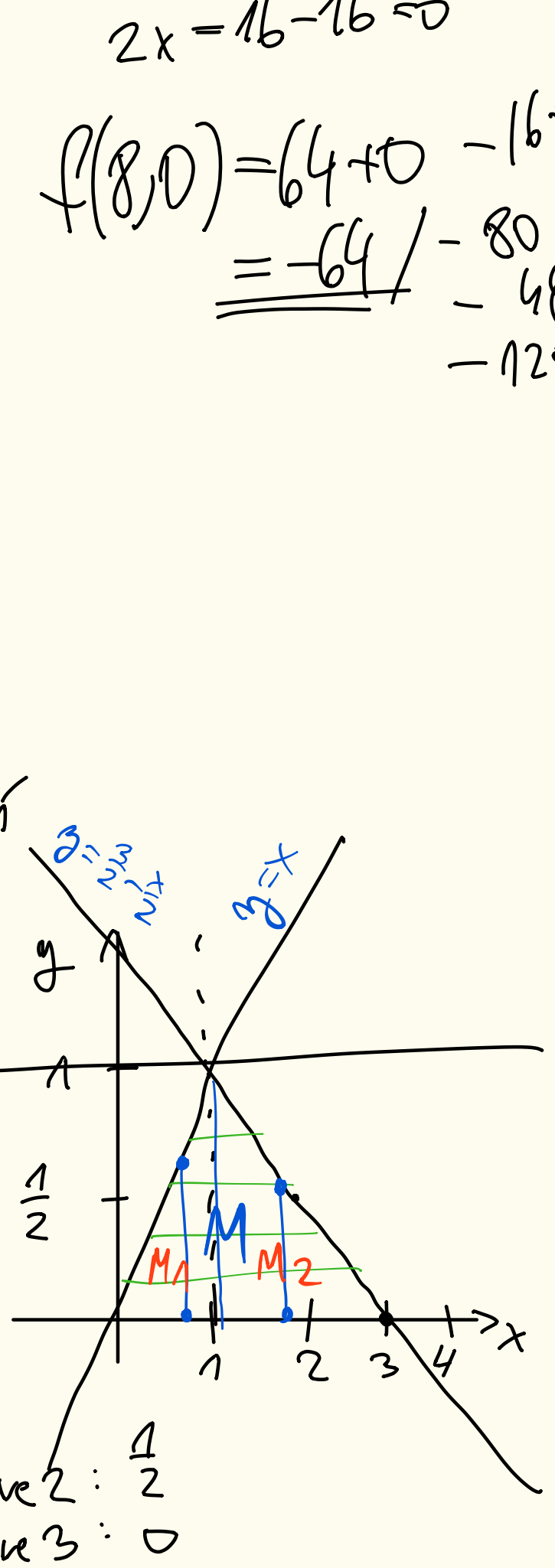
$Hf(-4,0) = \begin{pmatrix} 2 & 8 \\ 8 & 0 \end{pmatrix}$

$\Delta_1 = 2$

$\Delta_2 = -64$ indefinitní

sedlový bod

(3) $\int_0^1 \int_y^{3-2y} f(x,y)\,dx\,dy = \iint_{M_1} f\,dA + \iint_{M_2} f\,dA =$



M_1: $0 \le x \le 1$

$0 \le y \le x$

$x = 3 - 2y$

$2y = 3 - x$

$y = \frac{3}{2} - \frac{x}{2}$

M_2: $1 \le x \le 3$

$0 \le y \le \frac{3}{2} - \frac{x}{2}$

ve 2: $\frac{1}{2}$

ve 3: 0

$$\int_0^1 \int_0^x f(x/y)\,dy\,dx + \int_1^3 \int_0^{3/2-x/2} f(x/y)\,dy\,dx$$

$$\int S = \iint f\,dA$$

V polárních:

V: $0 \le \varphi \le \pi/4$

$0 \le r \le \frac{3}{2(\sin\varphi+\cos\varphi)}$

$$\int_0^{\pi/4} \int_0^{\frac{3}{2\sin\varphi+\cos\varphi}} f(r\cos\varphi, r\sin\varphi)\, r\,dr\,d\varphi$$

$y = \frac{3}{2} - \frac{x}{2}$

$r\sin\varphi = \frac{3}{2} - \frac{r\cos\varphi}{2}$

~~$r\sin\varphi + r\cos\varphi = \frac{3}{2}$~~

~~$r(\sin\varphi+\cos\varphi) = \frac{3}{2}$~~

$r = \frac{3}{2\sin\varphi+\cos\varphi}$

$2r\sin\varphi = 3 - r\cos\varphi$

$2r\sin\varphi + r\cos\varphi = 3$

$r(2\sin\varphi+\cos\varphi) = 3$

(4) $\int_C \vec{F}\cdot d\vec{S} = \int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dA$

C: $x^2+y^2 = y$

$\vec{F} = (y+e^x, 3x^2-5y)$

$x^2+y^2-y = 0$

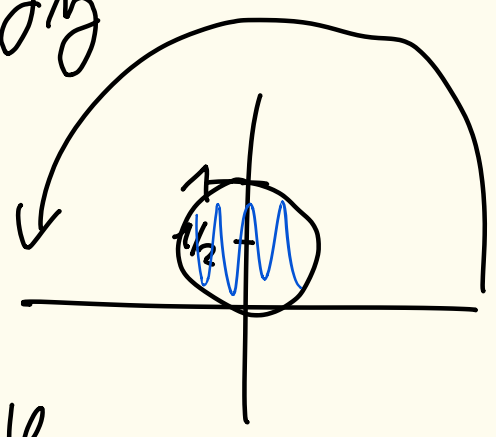
$x^2+\left(y-\frac{1}{2}\right)^2 = 1/4$

$r = \sqrt{1/4} = 1/2$

$\frac{\partial F_2}{\partial x} = 6x$

$\frac{\partial F_1}{\partial y} = 1$

$6x-1$



$x = r\cos\varphi$

$y = r\sin\varphi$

$r^2\cos^2\varphi + r^2\sin^2\varphi = r\sin\varphi$

$r^2(\cos^2\varphi + \sin^2\varphi) = r\sin\varphi$

$r^2 = r\sin\varphi$

$r = \sin\varphi$

D: $0 \le \varphi \le \pi$

$0 \le r \le \sin\varphi$

$6x-1 \le 6r\cos\varphi - 1$

$$\int_0^{\pi}\int_0^{\sin\varphi} 6r^2\cos\varphi - r \, dr\,d\varphi = \int_0^{\pi}\left[2r^3\cos\varphi - \frac{r^2}{2}\right]_{r=0}^{r=\sin\varphi} d\varphi =$$

$t=\sin\varphi$, $dt=\cos\varphi\,d\varphi$

$$= \int_0^{\pi} 2\sin^3\varphi\cos\varphi - \frac{\sin^2\varphi}{2}\, d\varphi = 2\int_0^0 t^3\,dt - \frac{1}{2}\int_0^{\pi}\sin^2\varphi\,d\varphi$$

$\sin^2\cdot 2\sin\varphi\cos\varphi$

$$-\frac{1}{2}\int_0^{\pi}\frac{1-\cos 2\varphi}{2}\,d\varphi = \underbrace{-\frac{1}{2}\int_0^{\pi}\frac{1}{2}\,d\varphi}_{0} + \underbrace{\frac{1}{2}\int_0^{\pi}\frac{\cos 2\varphi}{2}\,d\varphi}_{0}$$

$$-\frac{1}{2}\left[\frac{1}{2}\varphi\right]_0^{\pi} = -\frac{1}{4}\pi$$

(5.) V prvním oktantu objem tělesa E

$x^2+z^2\le 4 \quad z\ge 0 \quad y\ge 0$

$x+y+z\le 7$

Po parametrizaci V:

$y\ge 0$

$x^2+y^2\le 4 \quad z\ge 0$

$x+y+z\le 7$

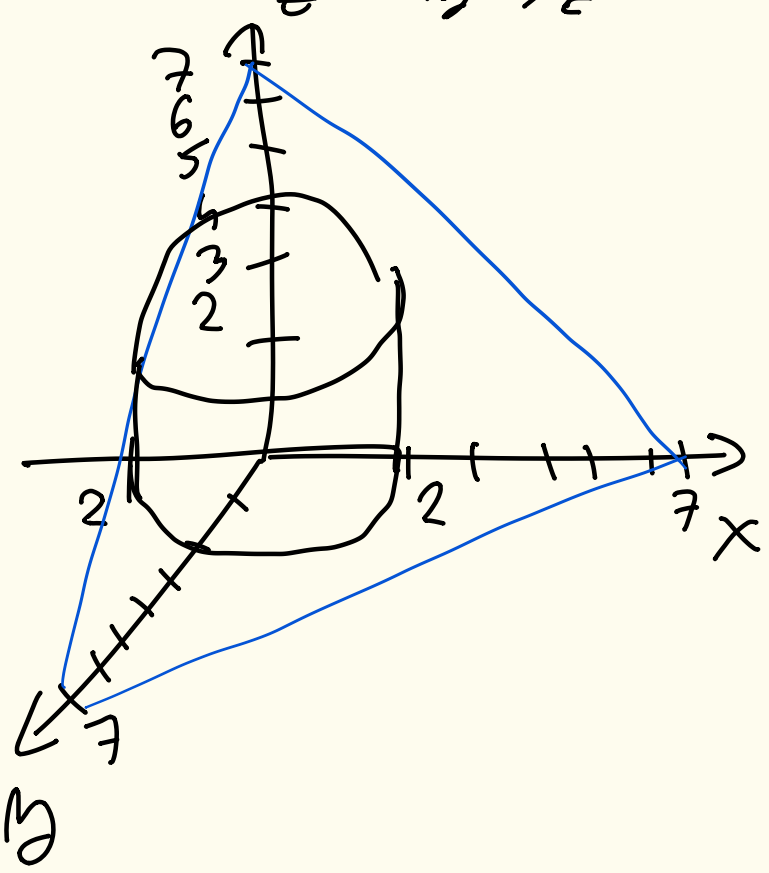
přes válcové souřadnice:

$x=r\cos\varphi$

$y=r\sin\varphi$

$h=z$

Parametrizace V: $x\to x$, $z\to y$, $y\to z$



$r\cos\varphi + r\sin\varphi + h = 7$

$h = 7-(r\cos\varphi + r\sin\varphi)$

$\pi\le\varphi\le 2\pi$

$0\le r\le 2$

$0\le h\le 7-(r\cos\varphi + r\sin\varphi)$

$$\int_\pi^{2\pi}\int_0^2\int_0^{7-(r\cos\varphi+r\sin\varphi)} 1\cdot r\,dh\,dr\,d\varphi = \int_\pi^{2\pi}\int_0^2 r\left[h\right]_{h=0}^{h=7-(r\cos\varphi+r\sin\varphi)} dr\,d\varphi =$$

$$= \int_\pi^{2\pi}\int_0^2 r(7-r\cos\varphi+r\sin\varphi)\,dr\,d\varphi = \int_\pi^{2\pi}\left[\frac{7r^2}{2}-\frac{r^3}{3}\cos\varphi+\frac{r^3}{3}\sin\varphi\right]_0^2 d\varphi$$

$7r - r^2\cos\varphi + r^2\sin\varphi$

$\frac{7r^2}{2} - \frac{r^3}{3}\cos\varphi + \frac{r^3}{3}\sin\varphi$

$$= \int_\pi^{2\pi} 14 - \frac{8}{3}\cos\varphi + \frac{8}{3}\sin\varphi\,d\varphi = 2\left[7\varphi\right]_{\varphi=\pi}^{\varphi=2\pi} = 2(14\pi-7\pi)$$

$$= 14\pi - \frac{16}{3}$$

$$= 2\int_\pi^{2\pi} 7 - \underbrace{\frac{4}{3}\cos\varphi}_{0} + \underbrace{\frac{4}{3}\sin\varphi}\,d\varphi$$

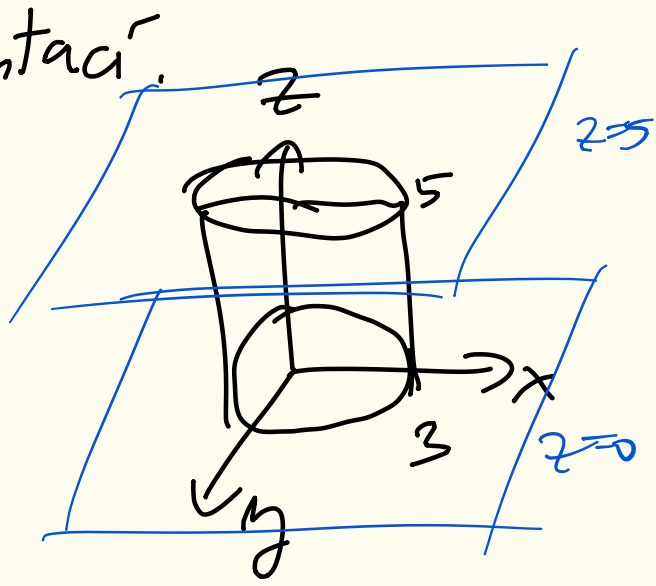
$$\frac{4}{3}\left[\cos\varphi\right]_\pi^{2\pi} = -\frac{4}{3}(1+1) = -\frac{8}{3}\cdot 2$$

(5) Užitím Gaussovy věty vypočítejte plošný int. $\iint_M \vec{F}\cdot d\vec{S}$, kde

$\vec{F} = (3x+y^2, y^3-z, -2z^2+x)$ Tah v.p.

a M je povrch omezeného útvaru s hranicí uvnitř plochami;

$x^2+y^2=9$, $z=0$, $z=5$ s vnější orientací.



$$\iint_M \vec{F}\cdot d\vec{S} = \iiint_T \operatorname{div} F\,dV$$

$$\operatorname{div} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3 + 3y^2 - 4z$$

$x = r\cos\varphi$
$y = r\sin\varphi$
$z = h$

$T: \quad 0 \le \varphi \le 2\pi$
$0 \le r \le 3$
$0 \le h \le 5$

$$\int_0^{2\pi}\int_0^{5}\int_0^{3} 3r + 3r^3\sin^2\varphi - 4hr \, dr\,dh\,d\varphi$$

$$3\int_0^{2\pi}\int_0^{5}\int_0^{3} r \, dr\,dh\,d\varphi + 3\int_0^{2\pi}\int_0^{5}\int_0^{3} r^3\sin^2\varphi \, dr\,dh\,d\varphi + \int_0^{2\pi}\int_0^{5}\int_0^{3} -4hr \, dr\,dh\,d\varphi$$

$$\frac{r^2}{2}\Big|_0^3 \cdot \frac{9h}{2}\Big|_0^5 \cdot \varphi\Big|_0^{2\pi}$$

$$3 \cdot \frac{45}{2} \cdot 2\pi$$

$$3 \cdot 45\pi$$

$$135\pi$$

$$3 \cdot \frac{81}{4} \cdot 5 \cdot \int_0^{2\pi} \sin^2\varphi \, d\varphi$$

$$\left[\frac{r^4}{4}\sin^2\varphi\right]_0^3 \cdot h\Big|_0^5 \cdot \int_0^{2\pi} \frac{1-\cos 2\varphi}{2} \, d\varphi$$

$$\frac{81}{4}\sin^2\varphi \cdot h\sin^2\varphi$$

$$\frac{1}{2}\int_0^{2\pi} 1 - \cos 2\varphi \, d\varphi$$

$$\pi \left[\varphi - \frac{\sin 2\varphi}{2}\right]_0^{2\pi}$$

$$2\pi$$

$$15\pi \cdot \frac{81}{4}$$

$$-4 \int_0^{2\pi}\int_0^{5}\int_0^{3} hr \, dr\,dh\,d\varphi$$

$$-4 \cdot \frac{9}{2} \cdot \frac{25}{2} \cdot 2\pi$$

$$-8\pi \cdot \frac{1}{4} \cdot 9 \cdot 25$$

$$-2 \cdot 9 \cdot 25\pi$$

$$-50 \cdot 9\pi$$

$$\left[\frac{hr^2}{2}\right]_{r=0}^{r=3} \cdot \frac{9h}{2}\Big|_0^5 \cdot \varphi\Big|_0^{2\pi}$$

$$\left[\frac{h^2}{2}\right]_0^5 \cdot \frac{25}{2}$$

$$135\pi + 15\pi \cdot \frac{81}{4} - 50 \cdot 9\pi$$

$$\frac{135 \cdot 4 + 15 \cdot 81 - 50 \cdot 9 \cdot 4}{4}\pi$$

$$\frac{540 + 1215 - 1800}{4}\pi$$

$$\frac{-45}{4}\pi$$

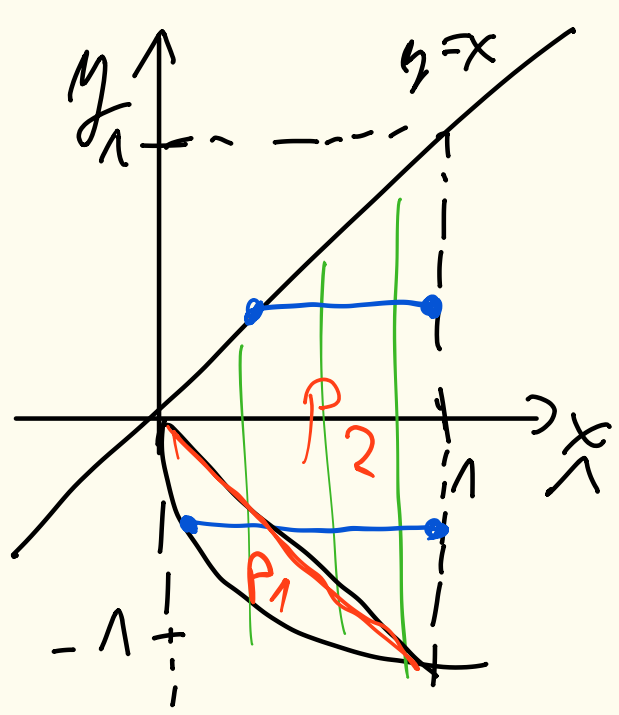
200 600
270
400 140
540

81
15
5
40
81
1215

1215
540
1755

450
900
1800

(3) $\int_0^1 \int_{-\sqrt{x}}^{x} f \, dy \, dx$



$$\int_{-1}^{0} \int_{y^2}^{1} f \, dx \, dy + \int_{0}^{1} \int_{y}^{1} f \, dx \, dy$$

v polárních

$$\int_{-\pi/4}^{-\pi/2} \int_{0}^{\frac{\cos\varphi}{\sin^2\varphi}} f(r\cos\varphi, r\sin\varphi)\, r \, dr \, d\varphi + \int_{-\frac{\pi}{4}}^{\pi/4} \int_{0}^{1/\cos\varphi} f(r\cos\varphi, r\sin\varphi)\, r \, dr \, d\varphi$$

$y = -\sqrt{x}$

$y^2 = x$

$r^2 \sin^2\varphi = r\cos\varphi$

$r \sin^2\varphi = \cos\varphi$

$x = 1$

$r\cos\varphi = 1$

$r = \frac{1}{\cos\varphi}$

$r = \frac{\cos\varphi}{\sin^2\varphi}$

Spočtěte integrál nehtarového pole

$$\vec{F} = (3y^2, 4z, -6x)$$

podél hranice C, která je okrajem plochy

$$M: x+y+z=1 \ \& \ x,y,z \geq 0$$

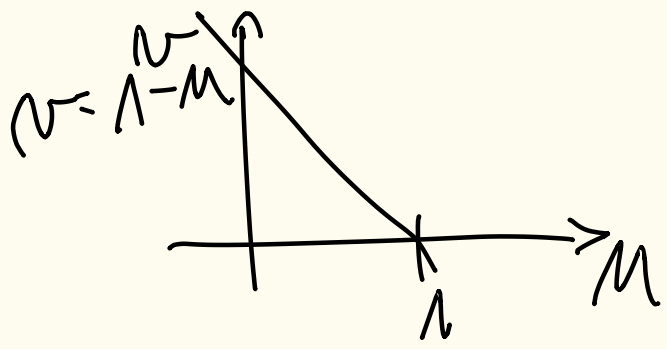
$$\operatorname{rot} F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y^2 & 4z & -6x \end{vmatrix} = e_1(0-4) + e_2(0+6) + e_3(0-3y)$$

$$= -4e_1 - 6e_2 - 3ye_3$$

$$z = 1-x-y$$

$$= (-4, -6, -6y)$$

Param. funkce $\varphi(u,v) = (u, v, 1-u-v)$



$u \in \langle 0,1 \rangle$

$v \in \langle 0, 1-u \rangle$

$$\frac{\partial \varphi}{\partial u} = (1,0,-1)$$

$$\frac{\partial \varphi}{\partial v} = (0,1,-1)$$

$$\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} =$$

$$= e_1(0+1) + e_2(+1) + e_3(1)$$

$$= (1,1,1)$$

$$\int_C F\,ds = \iint_M \operatorname{rot} F \cdot dS = \int_0^1 \int_0^{1-u} (-4, -6, -6v) \cdot (1,1,1)\,dv\,du$$

$$= \int_0^1 \int_0^M -10 - 6v \, dv\,du =$$

$$= \int_0^1 \left[-10u - 3u^2\right]_0^{1-u} du = \int_0^1 -10(1-u) - 3(1-u)^2 \, du$$

$$-10 + 10u - 3(1 - 2u + u^2)$$

$$-10 + 10u - 3 + 6u - 3u^2$$

$$-3u^2 + 16u - 13$$

$$\int_0^1 -3u^2 + 16u - 13 \, du$$

$$= \left[-u^3 + 8u^2 - 13u\right]_0^1 = -1 + 8 - 13 = \underline{\underline{-6}}$$

Green

$$\vec{F} = (e^x - x^2, 2xy)$$

$$\frac{\partial F_2}{\partial x} = 2y$$

$$\frac{\partial F_1}{\partial y} = 0$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

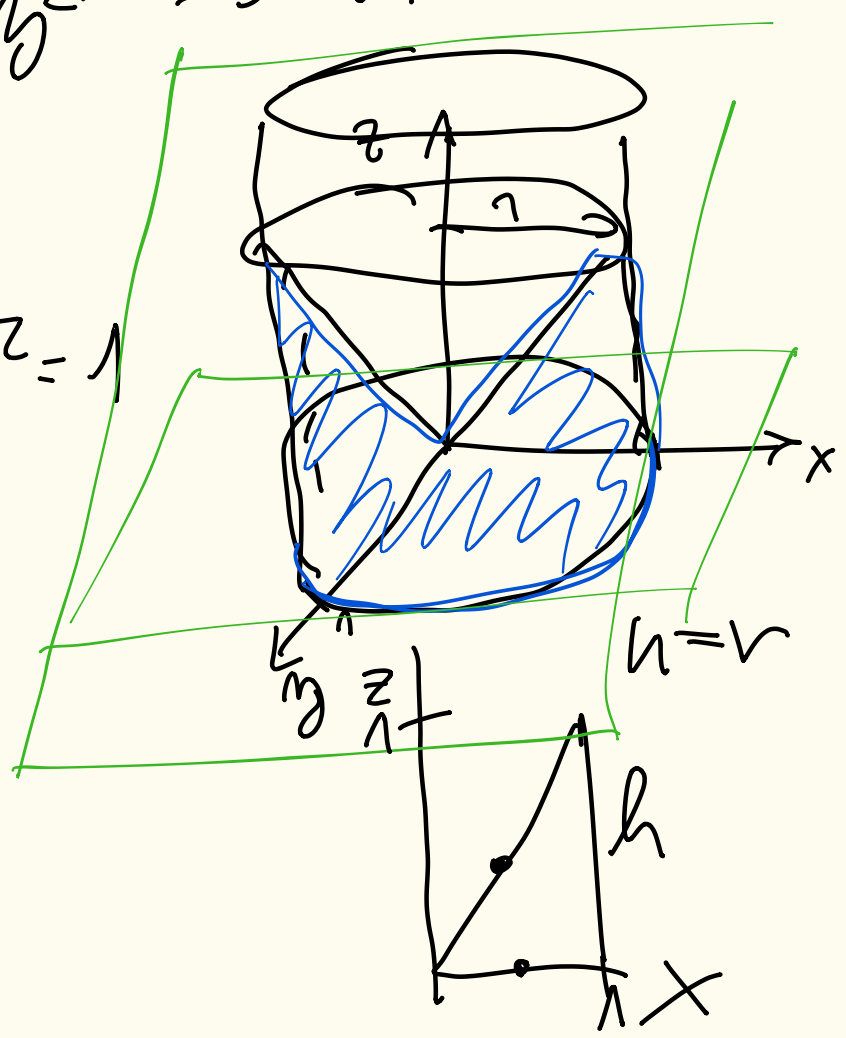
$$\oint_C F\,ds = \iint_D 2y \, dS = \int_0^1 \int_0^{\sqrt{1-x^2}+1} 2y$$

(5) E je těleso shora $z = \sqrt{x^2 + y^2}$ → kužel

zdola

$z = 0$

ze stran $x^2 + y^2 = 1$



$$\sigma(x,y,z) = y^2$$

$$0 \le \varphi \le 2\pi$$

$$0 \le h \le r$$

$$0 \le r \le 1$$

$$\int_0^{2\pi}\int_0^{1}\int_0^{r} r^3 \sin^2\varphi \, dh\, dr\, d\varphi =$$

$$= \int_0^{2\pi}\int_0^{1} r^4 \sin^2\varphi \, dr\, d\varphi = \frac{1}{5}\int_0^{2\pi} \sin^2\varphi \, d\varphi =$$

$$= \frac{1}{10}\int_0^{2\pi} 1-\cos 2\varphi \, d\varphi = \frac{1}{10}\left[\varphi - \frac{1}{2}\sin 2\varphi\right]_0^{2\pi}$$

$$= \frac{1}{10}\, 2\pi = \underline{\underline{\frac{\pi}{5}}}$$

(1.) $\lim_{(x,y)\to(0,0)} \dfrac{3xy^2}{x^2+y^2}$ $\quad x=0$

$$\frac{0}{y^2} \to 0$$

$y=kx$ $\qquad y=x$

$$\frac{3x^3k^2}{x^2+k^2x^2} = \frac{3xk^2x^2}{x^2(k+1)} = \frac{3x^3}{2x^2} = \frac{3}{2}x \xrightarrow{x\to 0} 0$$

$x\to 0$ $\quad 0$

$$\frac{3r\cos\varphi\, r^2\sin^2\varphi}{r^2(\sin^2\varphi+\cos^2\varphi)} = \left(3r\cos\varphi\sin^2\varphi\right) \to 0$$

ograniczona

$r\to 0$

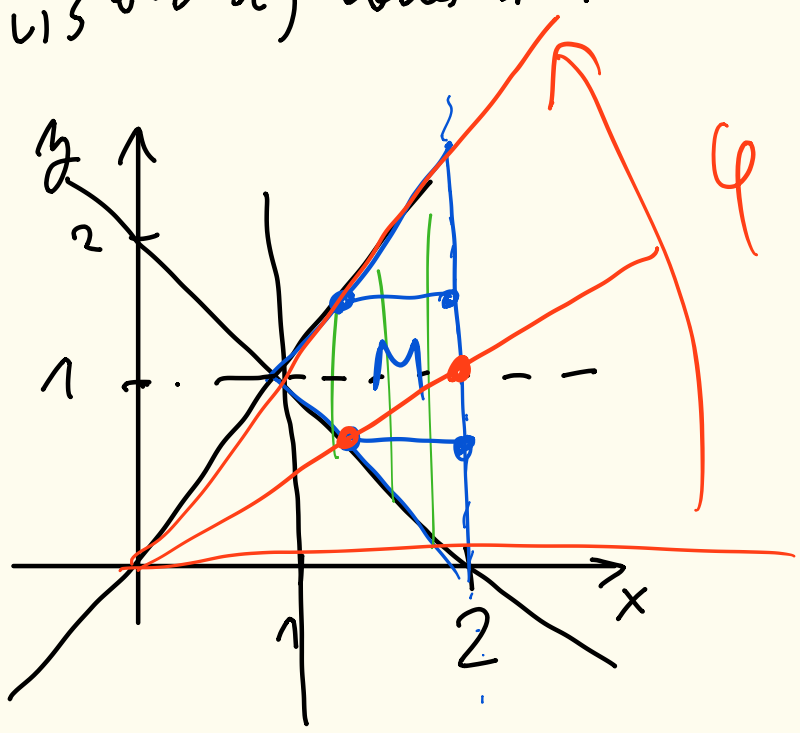
limita je 0

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \qquad y=kx$$

$$\frac{x^2-k^2x^2}{x^2+k^2x^2} \qquad \frac{x^2(1-k^2)}{x^2(1+k^2)} = \frac{1-k^2}{1+k^2}$$

zależy od k, nie ma limitu

(3) $$\int_1^2 \int_{2-x}^{x} f \, dy\,dx$$



$$\int_0^1 \int_1^2 f \, dx\,dy + \int_1^2 \int_1^2 f \, dx\,dy$$

$y = 2-x$ $\quad x=2$ $\quad r\cos\varphi = 2$

$$\int_{\pi/4}^{0} \int_{2/\cos\varphi}^{2/(\sin\varphi+\cos\varphi)} f(r\cos\varphi, r\sin\varphi)\, r\, dr\, d\varphi$$

$y = 2-x$

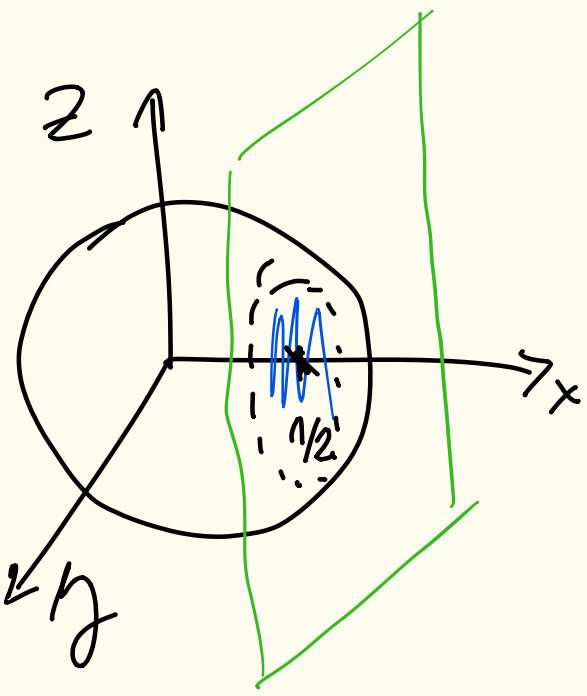
$r\sin\varphi = 2 - r\cos\varphi$

$r\sin\varphi + r\cos\varphi = 2$

(4) C:

$x^2+y^2+z^2=1$ & $x=1/2$

$y^2+z^2 = 3/4$



$f(x,y,z) = y^2+|z|$

$\varphi(t) = \left(\frac{\sqrt{3}}{2}\cos t, \frac{\sqrt{3}}{2}\sin t\right) \quad t\in\langle 0,\pi\rangle$

$\left\| -\frac{\sqrt{3}}{2}\sin t, \frac{\sqrt{3}}{2}\cos t \right\| = \frac{\sqrt{3}}{2}$

$$2\int_C \sigma \, ds = 2\int_0^{\pi} \left(\frac{3}{4}\cos^2 t + \frac{\sqrt{3}}{2}\sin t\right) \cdot \|\varphi'(t)\| =$$

$$= \sqrt{3}\int_0^{\pi} \frac{3}{4}\cos^2 t + \frac{\sqrt{3}}{2}\sin t \, dt =$$

$\cos^2 t \to \frac{1+\cos 2t}{2}$

$$= \sqrt{3}\left[\frac{3}{4}\cdot\left(\frac{1}{2}t + \frac{\sin 2t}{4}\right) - \frac{\sqrt{3}}{2}\cos t\right]_0^{\pi}$$

$$= \sqrt{3}\left(\frac{3}{4}\frac{\pi}{2} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = \sqrt{3}\cdot\frac{3\pi}{8} =$$

$$= \frac{\sqrt{3}\cdot 3\pi}{8}$$

Pomocí Gaussovy věty zjistěte, jakým je tok pole

$$\vec{F}(x,y,z) = (z^2 - x, -xy + y, 3z - y)$$

$z = 4 - y^2 \quad z^2 + y^2 = 4$

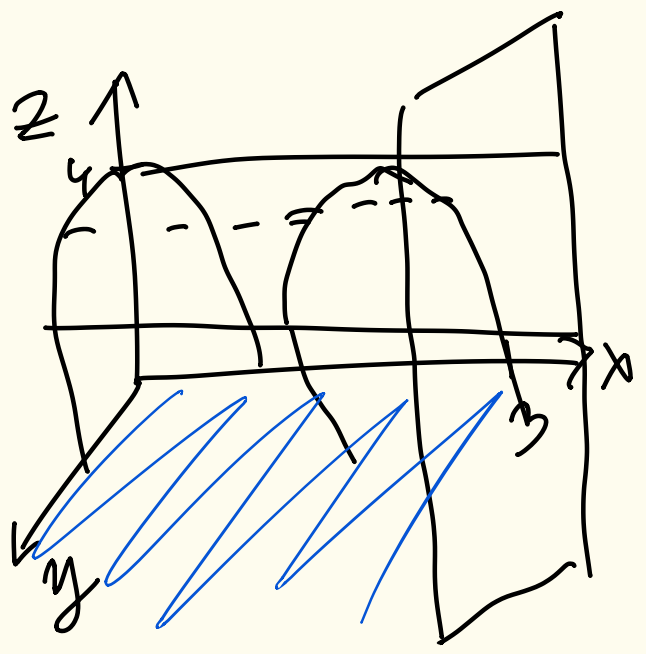
$z = 0$

$x = 0$

$x = 3$

$y = \pm 2$

$z^2 = 4 - y^2$



$$\int_M F \, dS = \iiint_T \operatorname{div} F \, dV$$

$0 \le x \le 3$

$-2 \le y \le 2$

$0 \le z \le 4 - y^2$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = -1 - x + 3$$

$$\int_0^3 \int_{-2}^{2} \int_0^{4-y^2} 2-x \, dz\, dy\, dx$$

$2(4-y^2) - x(4-y^2)$

$8 - 2y^2 - 4x + xy^2$

$$\int_0^3 \int_{-2}^{2} \Big[2z - xz\Big]_{z=0}^{z=4-y^2} dy\, dx = \int_0^3 \int_{-2}^{2} 8 - 2y^2 - 4x + xy^2 \, dy\, dx$$

$$\int_0^3 \Big[8y - \frac{2y^3}{3} - 4xy + \frac{xy^3}{3}\Big]_{y=-2}^{y=2} dx$$

$$\frac{16}{3} - 8x + \frac{8x}{3} + 16 - \frac{16}{3} - 8x + \frac{8x}{3}$$

$$= 32 - \frac{32}{3} - 16x + \frac{16x}{3}$$

$$= \frac{96 - 32 - 48x + 16x}{3} = \frac{64}{3} - \frac{32x}{3}$$

$$\int_0^3 \frac{64}{3} - \frac{32x}{3} \, dx = \Big[\frac{64x}{3} - \frac{16x^2}{3}\Big]_{x=0}^{x=3} \qquad 16 \cdot 3$$

$$= 64 - 48 = 16$$

(4) Ukažte, že vektorové pole

$$\vec{F} = (2xz + \sin y,\ x\cos y,\ x^2) \text{ v } \mathbb{R}^3 \qquad \int_C \vec{F}\, ds$$

$$\mathrm{rot}\, F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz + \sin y & x\cos y & x^2 \end{vmatrix} = (0,0,0)$$

$2x - 2x - \cos y + \cos y$

je potenciálové (konzervativní)

Potenciál

$$\int_a^b F(\varphi(t)) \cdot \varphi'(t)\, dt \qquad t \in \langle a, b \rangle$$

pole vekt. pole

pokud křivka C leží v rovině majíc x z a stejné úsečky

$$\frac{\partial f}{\partial x} = 2xz + \sin y \quad (1)$$

$$\frac{\partial f}{\partial y} = x\cos y \quad (2)$$

$$\frac{\partial f}{\partial z} = x^2 \quad (3)$$

$f = zx^2 + C_3(x,y)$ int (3) podle z

$\frac{\partial f}{\partial y} = 0 + \frac{\partial C_3}{\partial y} = x\cos y$ ∂ podle y

$C_3 = x\sin y + C_2(x)$ int podle y

$f(x,y,z) = zx^2 + x\sin y + C_2$

$\frac{\partial f}{\partial x} = 2xz + \sin y + C_2' = 2xz + \sin y$

$C_2' = 0$

$C_2 = k$

$f(x,y,z) = zx^2 + x\sin y + k$

Jelikož je pole potenciální, hodnota $\int_C \vec{F}\,ds$ nezávisí na cestě. Pouze na počátečním bodě.

$f(A) - f(A) = 0$!

(1) najdeme rovnici tečné roviny

$$x^2yz + 3y^2 - 2xz^2 + 8z = 0$$

$$\frac{\partial f}{\partial x} = 2xyz - 2z^2$$

$$\frac{\partial f}{\partial y} = x^2z + 6y$$

$$\frac{\partial f}{\partial z} = x^2y - 4xz + 8$$

$$\vec{n} = \nabla f(x,y,z) = (2xyz - 2z^2,\ x^2z + 6y,\ x^2y - 4xz + 8)$$

$$2 + 4 + 8$$

$$\nabla f(1,2,-1) = (-6, 11, 14)$$

$$x = 1 - 6t$$
$$y = 2 + 11t \qquad t \in \mathbb{R}$$
$$z = -1 + 14t$$

kde průsečík prochází rovinou $4x + y + z = 4$

$$4(1-6t) + 2 + 11t - 1 + 14t = 4$$

$$4 - 24t + 2 + 11t - 1 + 14t = 4$$

$$t = -1$$

$$x = 1 + 6 = 7$$
$$y = 2 - 11 = -9$$
$$z = -1 - 14 = -15$$

$$A = (7, -9, -15)$$

(2.) $\nabla f(x,y) = \left(2xe^{x^2+2y^2} - \sin(x-2y),\ 4ye^{x^2+2y^2} + 2\sin(x-2y)\right)$

$$Hf_{(x,y)} = \begin{pmatrix} 2e^{x^2+2y^2} + 4x^2e^{x^2+2y^2} - \cos(x-2y) & 8xye^{x^2+2y^2} + 2\cos(x-2y) \\ 8xye^{x^2+2y^2} + 2\cos(x-2y) & 4e^{x^2+2y^2} + 16y^2e^{x^2+2y^2} - 4\cos(x-2y) \end{pmatrix}$$

$2 + 0 - 1$

$0 + 2$

$4 + 0 - 4$

$\nabla f(0,0) = (0,0)$

$f(0,0) = 2$

$$Hf_{(0,0)} = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \quad \begin{matrix} \Delta_1 = 1 > 0 \\ \Delta_2 = -4 < 0 \end{matrix} \quad \begin{matrix} \text{indefinitní} \\ \text{sedlový bod} \end{matrix}$$

$$T_2(a_0+\vec{h}) = f(a_0) + f'(a_0)\vec{h} + \frac{1}{2} H_{f(a_0)}(\vec{h},\vec{h})$$

$$= 2 + 0 + \frac{(x\ y)\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}{2}$$

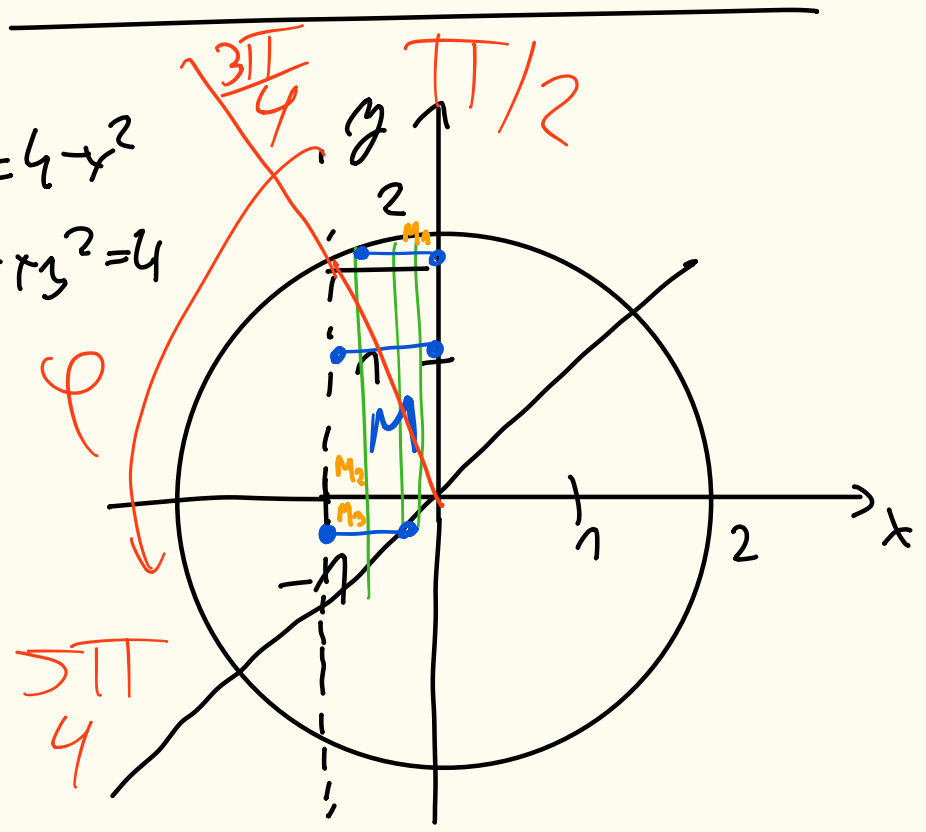
$$= 2 + \frac{(x\ y)\begin{pmatrix} x+2y \\ 2x \end{pmatrix}}{2} = 2 + \frac{x^2+2xy+2xy}{2}$$

(3) Prepište integrál

$$\int_{-1}^{0} \int_{x}^{\sqrt{4-x^2}} f\,dy\,dx$$

$y^2 = 4 - x^2$

$x^2 + y^2 = 4$



$U_1: \sqrt{4-x^2} \le x \le 0$

$\sqrt{3} \le y \le 2$

$U_2: -1 \le x \le 0$

$0 \le y \le \sqrt{3}$

$U_3: -1 \le x \le y$

$-1 \le y \le 0$

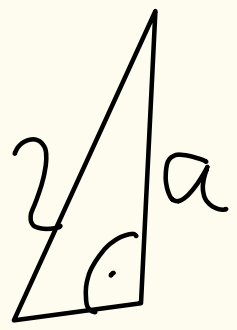
$M = M_1 \cup M_2 \cup M_3$

$x^2+y^2=4 \qquad x=y$

$1+y^2=4$

$y^2=3$

$y=\sqrt{3}$



$1+a^2=4$

$a^2=3 \quad a=\sqrt{3}$

$$= \iint_M f\,dA = \int_{\sqrt{3}}^{2}\int_{0}^{\sqrt{4-x^2}} f\,dx\,dy + \int_{0}^{\sqrt{3}}\int_{0}^{y} f\,dx\,dy + \int_{-1}^{0}\int_{-1}^{y} f\,dx\,dy$$

$P_1: \frac{\pi}{2} \le \varphi \le \frac{3\pi}{4}$

$0 \le r \le 2$

$x^2+y^2=4$

$r^2\cos^2\varphi + r^2\sin^2\varphi = 4$

$r^2 = 4$

$r = 2$

$P_2: \frac{3\pi}{4} \le \varphi \le \frac{5\pi}{4}$

$0 \le r \le -\frac{1}{\cos\varphi}$

$x=-1$

$r\cos\varphi = -1$

$r = -\frac{1}{\cos\varphi}$

$$\iint_M f\,dA = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\int_{0}^{2} f(r\cos\varphi, r\sin\varphi)\,r\,dr\,d\varphi + \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}}\int_{0}^{-1/\cos\varphi} f(r\cos\varphi, r\sin\varphi)\,r\,dr\,d\varphi$$

(4) $\vec{F} = (6xz, \cos y, 3x^2+y)$

$\vec{G} = (-e^x, 0, 2z-y)$

$\vec{H} = (6xz-e^x, \cos y, 3x^2+2z)$

$$\operatorname{rot} F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xz & \cos y & 3x^2+y \end{vmatrix} = 1\cdot e_1 \rightarrow \text{není konzervativní}$$

$\parallel (1,0,0)$

$$\operatorname{rot} G = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -e^x & 0 & 2z-y \end{vmatrix} = e_1(-1-0) + 0e_2 + 0e_3)$$

$(-1,0,0)$

$$\operatorname{rot} H = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xz-e^x & \cos y & 3x^2+2z \end{vmatrix} = 0e_1 + 0e_2 + 0e_3$$

$= (0,0,0)$ je konzervativní

$\Downarrow$

má potenciál

$\frac{\partial h}{\partial x} = 6xz - e^x$

$\frac{\partial h}{\partial y} = \cos y$

$\frac{\partial h}{\partial z} = 3x^2 + 2z$

$h = \sin y + C_2(x,z)$

$\frac{\partial h}{\partial z} = \frac{\partial C_2}{\partial z} = 3x^2 + 2z$

$C_2 = 3x^2 z + z^2 + C_1(x)$

$f(x,y,z) = -\sin y + 3x^2 z + z^2 + C_1(x)$

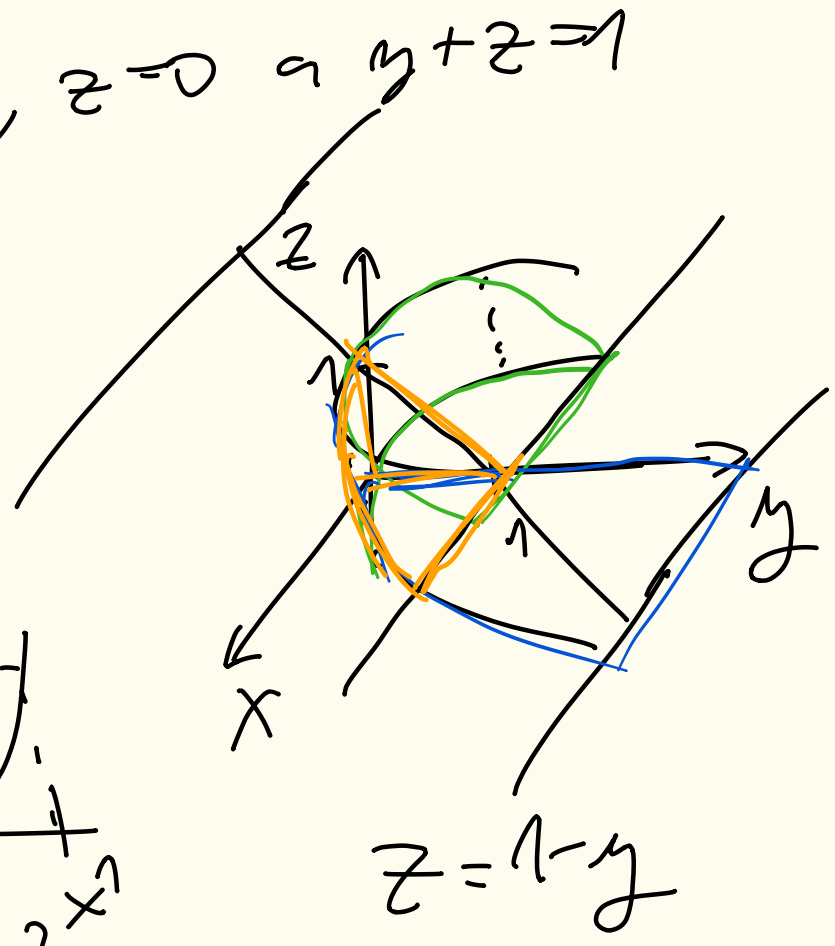
$\frac{\partial f}{\partial x} = 6xz + C_1' = 6xz - e^x$

$C_1 = -e^x + k$

$f(x,y,z) = \sin y + 3x^2 z + z^2 - e^x + k$

(5) V? rovinnou oblastí

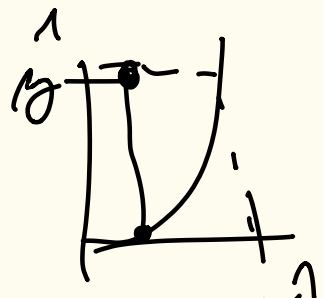
válec $y=x^2$ a rovinami $x=0$, $z=0$ a $y+z=1$



$0 \le x \le 1$

$x^2 \le y \le 1$

$0 \le z \le 1-y$



$$\int_0^1 \int_{x^2}^1 \int_0^{1-y} 1 \, dz\,dy\,dx$$

$$\int_0^1 \int_{x^2}^1 1-y \, dy\,dx = \int_0^1 \left[y - \frac{y^2}{2}\right]_{x^2}^1 dx = \int_0^1 1 - x^2 - \frac{1}{2} + \frac{x^4}{2} \, dx$$

$$= \int_0^1 \frac{1}{2} - x^2 + \frac{x^4}{2} \, dx = \left[\frac{x}{2} - \frac{x^3}{3} + \frac{x^5}{10}\right]_0^1 = \frac{1}{2} - \frac{1}{3} + \frac{1}{10} =$$

$$= \frac{15-10+3}{30} = \frac{8}{30}$$

$$y = \frac{3-x}{2}$$

$$2y = 3 - x$$

$$2r\sin\varphi = 3 - r\cos\varphi$$

$$2r\sin\varphi + r\cos\varphi = 3$$

$$r(2\sin\varphi + \cos\varphi) = 3$$

$$r = \frac{3}{2\sin\varphi + \cos\varphi}$$

$$0 \le \varphi \le \frac{\pi}{4}$$

$$\int_0^{\pi/4} \int_0^{\frac{3}{2\sin\varphi+\cos\varphi}} r \, dr\,d\varphi$$

(1)

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^3}{x^2+y^2}$$

$x=0$

$$\frac{y^3}{y^2} = y \xrightarrow{y\to 0} 0$$

$y=kx$

$$\frac{x^2+k^3x^3}{x^2+k^2x^2} = \frac{x^2(1+k^3x)}{x^2(k^2+1)} = \frac{1}{k^2+1} \xrightarrow{x\to 0} \text{nemá limitu}$$

$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$$

$x=0$

$$\frac{y^2}{y^2} = 1$$

$y=kx$

$$\frac{(x+kx)^2}{x^2+k^2x^2} = \frac{x^2+2x^2k+k^2x^2}{x^2+k^2x^2} = \frac{x^2(1+2k+k^2)}{x^2(1+k^2)} =$$

$$= \frac{1+2k+k^2}{1+k^2} \longrightarrow \text{nemá limitu}$$

(2.) Nad libovolným bodem elipsy $\frac{x^2}{4}+\frac{y^2}{9}\le 1$ dosahuje výraz

$z=2x+y$ největší výšky.

Rozložím na M^0 a ∂M

∂M: $g(x,y) = \frac{x^2}{4}+\frac{y^2}{9}-1$

$f(x,y) = 2x+y$

$$h(x,y) = 2x+y-\lambda\left(\frac{x^2}{4}+\frac{y^2}{9}-1\right)$$

$$\frac{\partial h}{\partial x} = 2-\frac{\lambda x}{2} = 0 \qquad 2=\frac{\lambda x}{2} \qquad 4=\lambda x$$

$$\frac{\partial h}{\partial y} = 1-\frac{2\lambda y}{9} = 0 \qquad 1=\frac{2\lambda y}{9} \qquad \frac{9}{2}=\lambda y$$

$$\frac{\partial h}{\partial \lambda} = -\frac{x^2}{4}-\frac{y^2}{9}+1 = 0$$

$$\frac{4\cdot 2}{9} = \frac{x}{y}$$

$$\frac{8}{9} = \frac{x}{y}$$

$$8y = 9x$$

$$y=\frac{9}{8}x \qquad x=\frac{8}{9}y$$

$$\frac{x^2}{4}+\frac{z^2}{9}-1=0$$

$$\frac{x^2}{4}+\frac{9^2}{8^2 9}x^2-1=0$$

$$\frac{x^2}{4}+\frac{9x^2}{64}-1=0$$

$$\frac{16x^2+9x^2}{64}-1=0$$

$$\frac{25x^2}{64}=1$$

$$25x^2=64 \qquad y=\frac{9}{8}\cdot\frac{8}{5}=\frac{9}{5}$$

$$x^2=\frac{64}{25}$$

$$x=\pm 8/5 \qquad y=\pm 9/5$$

Dva pode zidé $\int y\,dx$:

$$f(8/5, 9/5)=\frac{16}{5}+\frac{9}{5}=5 \longrightarrow \text{maximum}$$

$$f(-8/5, -9/5)=-5 \longrightarrow \text{minimum}$$

NO: nutná podmínka existuce extrému:

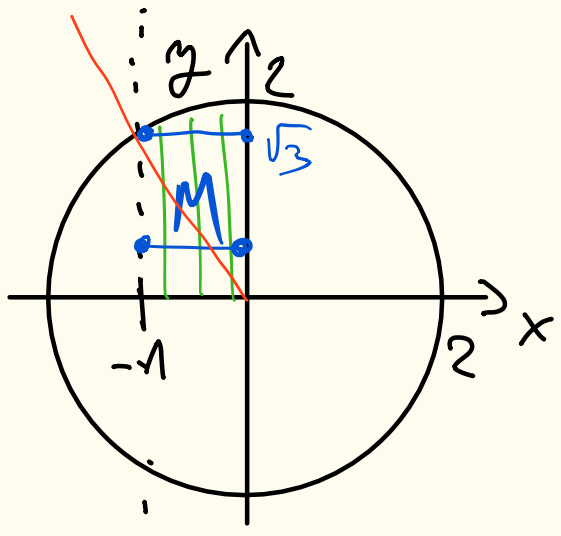
$\nabla f(x,y)=(2,1)$ ⟶ nikdy nenulová, neexistuje extrém

(3) $\int_{-1}^{0}\int_{0}^{\sqrt{4-x^2}} f\,dy\,dx$

$y=\sqrt{4-x^2}$

$y^2=4-x^2$

$x^2+y^2=4$



$$\int_{0}^{\sqrt{3}}\int_{-1}^{0} f\,dx\,dy+\int_{\sqrt{3}}^{2}\int_{0}^{\sqrt{4-y^2}} f\,dx\,dy$$

$x^2+y^2=4$

$1+y^2=4$

$y=\sqrt{3}$

$x=\sqrt{4-y^2}$

$\cos\alpha=\frac{1}{2}$

$\alpha=\pi/3$

$\frac{\pi}{2}-\pi/3=$

$\frac{3\pi}{6}-\frac{2\pi}{6}=\pi/6$

Polárně:

$$\int_{\pi/2}^{4\pi/6}\int_{0}^{2} f(r\cos\varphi, r\sin\varphi)\,r\,dr\,d\varphi+\int_{4\pi/6}^{\pi}\int_{0}^{-\frac{1}{r\cos\varphi}} f(r\cos\varphi, r\sin\varphi)\,r\,dr\,d\varphi$$

$x=-1 \qquad r\cos\varphi=-1$

$r=\frac{-1}{r\cos\varphi}$

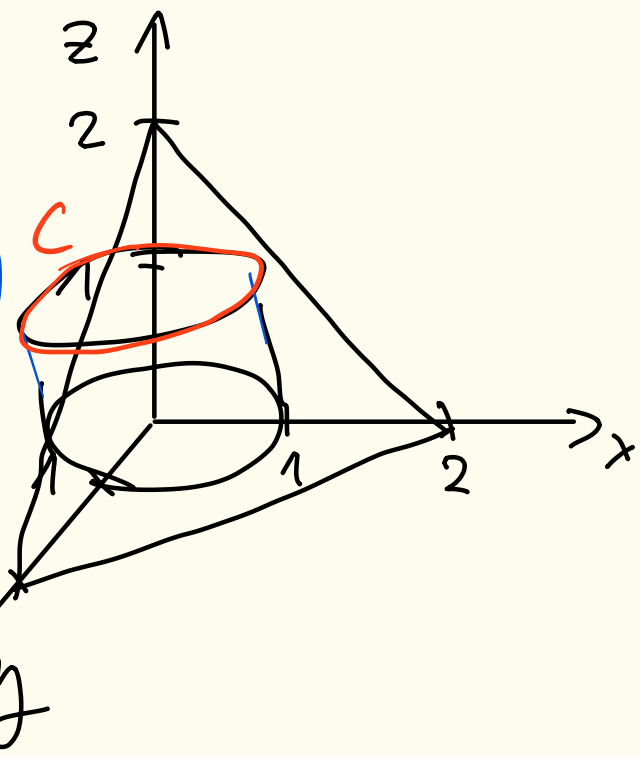
(4) Zparametrizujte křivku

$$C: x^2+y^2=1 \quad \& \quad x+y+z=2$$

jejíž orientace je kladná při průmětu do roviny xy. Spočítejte práci vektorového pole $\vec{F}(y,z,x)$ podél této křivky.

$$\varphi(t)=(\cos t, \sin t, 2-\cos t-\sin t)$$

$$t\in(0,2\pi)$$



práce vektorového pole

$$\int_C F\cdot ds = \int_0^{2\pi} F(\varphi(t))\cdot\varphi'(t)\,dt$$

$(-\sin t,\cos t,\sin t-\cos t)$

$$(\sin t, 2-\cos t-\sin t, \cos t)(-\sin t,\cos t,\sin t-\cos t)$$

$$-\sin^2 t+2\cos t-\cos^2 t-\sin t\cos t+\sin t\cos t-\cos^2 t$$

$$-1(\sin^2 t+\cos^2 t)+2\cos t-\cos^2 t$$

$$-1+2\cos t-\cos^2 t$$

$$\int_0^{2\pi} -1+2\cos t-\cos^2 t\,dt = -2\pi+0-\pi = \underline{\underline{-3\pi}}$$

$2\sin t$

$\frac{\sin 2t}{4}$

$$-\int \frac{1+\cos 2t}{2}$$

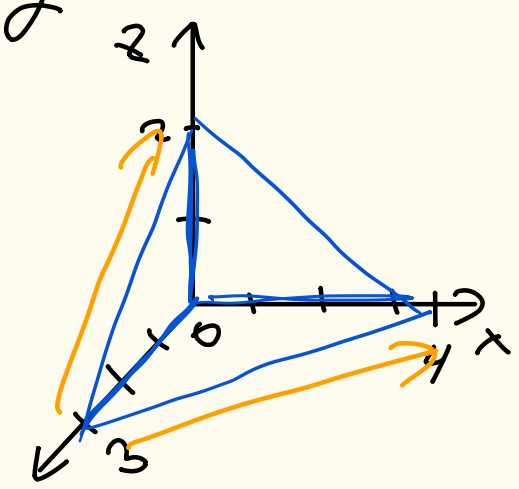
(5.) Užitím gaussovy věty vypočtěte plošný integrál

$$\iint_S \vec{F}\,d\vec{S}, \quad \text{kde } \vec{F} = (3xy, y^2, -x^2y^4)$$

(S)

S je povrch čtyřstěnu s vrcholy $(0,0,0), (0,0,2), (0,3,0), (4,0,0)$

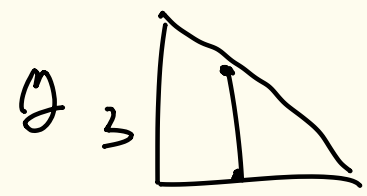
$$\operatorname{div} F = 3y + 2y = 5y$$



$$= \iiint_T \operatorname{div} F\,dS$$

$$= \int_0^4 \int_0^{3-\frac{3x}{4}} \int_0^{\frac{-4y-3x}{2}} \operatorname{div} F\,dz\,dy\,dx$$

rovina:



$(0,0,2) - (0,3,0) = (0,-3,2)$

$= (4,-3,0)$

$$\vec{n} \times \vec{v} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & -3 & 2 \\ 4 & -3 & 0 \end{vmatrix} = 6e_1, 8e_2, 12e_3$$

$(6,8,12) = \underline{\underline{(3,4,6)}}$

T: $0 \le x \le 4$

$0 \le y \le 3 - \frac{3}{4}x$

$0 \le z \le 3 - \frac{3}{4}x - \frac{4}{3}y$

$z = \dfrac{12 - 3x - 4y}{6}$

vše rovina: $3x + 4y + 6z - 12 = 0$

$$\int_0^4 \int_0^{3-\frac{3x}{4}} \int_0^{\frac{12-3x-4y}{6}} 5y \,dz\,dy\,dx = \int_0^4 \int_0^{3-\frac{3x}{4}} 5y \int_0^{\frac{12-3x-4y}{6}} 1 \,dz\,dy\,dx =$$

$$= \int_0^4 \int_0^{3-\frac{3x}{4}} 5y \cdot \frac{12-3x-4y}{6}\,dy\,dx =$$

$$= \int_0^4 \int_0^{3-\frac{3x}{4}} \frac{60y - 15xy - 20y^2}{6} \, dy\,dx =$$

$$\int_0^{3-\frac{3x}{4}} 10y - \frac{5xy}{2} - \frac{10y^2}{3}$$

$$y^2 = 9 - \frac{9x}{2} + \frac{9x^2}{16}$$

$$y^3 = 27 - \frac{81x}{4} + \frac{81x^2}{16} - \frac{27x^3}{64}$$

$$= \int_0^4 \left[5y^2 - \frac{5xy^2}{4} - \frac{10y^3}{9} \right]_{y=0}^{y=3-\frac{3x}{4}} dx =$$

$$= \int_0^4 5\left(9 - \frac{9x}{2} + \frac{9x^2}{16}\right) - \frac{5x\left(9 - \frac{9x}{2} + \frac{9x^2}{16}\right)}{4} - \frac{10\left(27 - \frac{81x}{4} + \frac{81x^2}{16} - \frac{27x^3}{64}\right)}{9} dx$$

$$= \int_0^4 5 \cdot 9 - \frac{5 \cdot 9x}{2} + \frac{5 \cdot 9x^2}{16} - \frac{5 \cdot 9x}{4} + \frac{5 \cdot 9x^2}{8} - \frac{5 \cdot 9x^2}{64}$$

$$- \frac{270}{9} + \frac{9x}{4} - \frac{9x^2}{16} + \frac{3x^3}{64}$$

$$= \left[45x - \frac{45x^2}{4} + \frac{15x^3}{16} - \frac{45x^2}{8} + \frac{15x^3}{8} - \frac{45x^4}{4 \cdot 64} - \right.$$

$$\left. \frac{270x}{9} + \frac{810x^2}{9 \cdot 8} - \frac{810x^3}{9 \cdot 3 \cdot 16} + \frac{270x^4}{9 \cdot 4 \cdot 64} \right|$$

$$= \underline{\underline{15}}$$

$$\left[45 - 30 \right] = \underline{\underline{15}}$$

Stokes $\vec{F} = (-y, \operatorname{arctg} z, \ln(1+x^2z^2))$

$$\operatorname{rot} F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & \operatorname{arctg} z & \ln(1+x^2z^2) \end{vmatrix} = -\frac{1}{1+x^2} e_1 - \frac{2xz^2}{1+x^2z^2} e_2 + 1 e_3$$

$$\left(-\frac{1}{1+z^2}, -\frac{2xz^2}{1+x^2z^2}, 1\right)$$

hvidun

(1) $\lim_{(x,y)\to(0,0)} \frac{x\cos y}{\sqrt{x^2+y^2}}$ $x=0$ $\frac{0}{y} \to 0$

$y=kx$ $\frac{x\cos kx}{\sqrt{x^2+k^2x^2}} = \frac{x\cos kx}{x\sqrt{1+k^2}} \simeq \frac{\cos kx}{\sqrt{1+k^2}} = \frac{1}{\sqrt{1+k^2}}$

$x^2(1+k^2)$

závisí na k $\to$ nemá limitu

$\lim_{(x,y)\to(0,0)} \frac{xy}{x^4+y^4}$ $x=0$ $\frac{0}{y^4} = 0$

$y=kx$ $\frac{x\cdot kx}{x^4+k^4x^4} = \frac{x^2k}{x^4(1+k^4)} = \frac{k}{x^2(1+k^4)} \underset{x\to 0}{=} +\infty$ nemá limitu

(2) kritické body $f(x,y) = x^2 + xy^2 - 16x$

nutná podmínka existence extrému

$\nabla f(x,y) = (2x + y^2 - 16, 2xy) = (0,0)$

$2x + y^2 - 16 = 0$

$2xy = 0$

$$2x = 16 - y^2$$

$$(16 - y^2)\,y = 0$$

$y = 0 \vee y = \pm 4$

$x = 8 \qquad x = 0$

body podezřelé z extrému

$(8, 0)$

$(0, 4)$

$(0, -4)$

$$Hf(x,y) = \begin{pmatrix} 2 & 2y \\ 2y & 2x \end{pmatrix}$$

$$Hf(8,0) = \begin{pmatrix} 2 & 0 \\ 0 & 16 \end{pmatrix} \quad \begin{matrix} \Delta_1 = 2 > 0 \\ \Delta_2 = 32 > 0 \end{matrix}$$

matice je positivně definitní (SÚ v. hnit)

lokální minimum

$$Hf(0,4) = \begin{pmatrix} 2 & 8 \\ 8 & 0 \end{pmatrix} \quad \begin{matrix} \Delta_1 = 2 > 0 \\ \Delta_2 = -64 < 0 \end{matrix}$$

matice je indefinitní (SÚ v.)

sedlový bod

$$Hf(0,-4) = \begin{pmatrix} 2 & -8 \\ -8 & 0 \end{pmatrix} \quad \begin{matrix} \Delta_1 = 2 > 0 \\ \Delta_2 = -64 > 0 \end{matrix}$$

matice je indefinitní (SÚ v.)

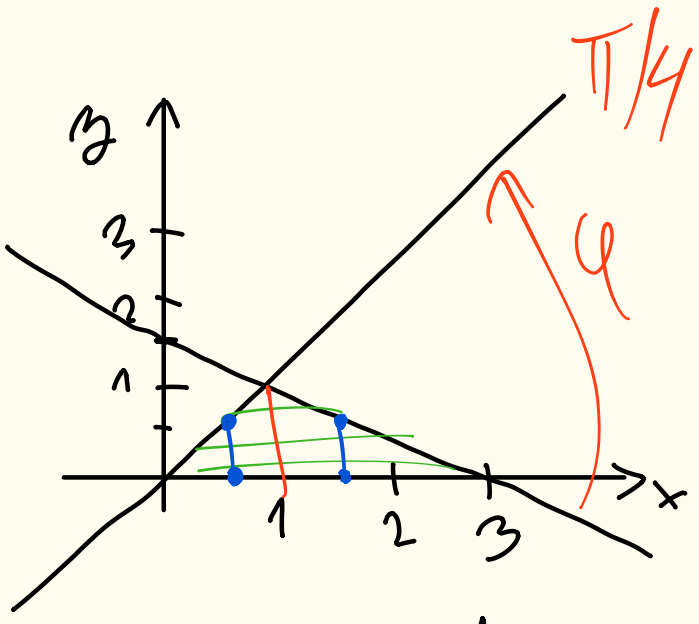
sedlový bod

(3) $\int_0^1 \int_y^{3-2y} f \, dx\,dy$

$x = 3 - 2y$

$2y = 3 - x$

$y = \frac{3-x}{2}$



(cartesian $y-x$:

$$\int_0^1 \int_0^x f \, dy\,dx + \int_1^3 \int_0^{\frac{3-x}{2}} f \, dy\,dx$$

$V_1 \quad 0 \le x \le 1$

$0 \le y \le x$

$V_2 \quad 1 \le x \le 3$

$0 \le y \le \frac{3-x}{2}$

Polar coords:

$$\int_0^{\pi/4}\int_0^{3/(\cos\varphi+2\sin\varphi)} f(r\cos\varphi, r\sin\varphi)\, r\, dr\, d\varphi$$

$x = 3-2y$

$r\cos\varphi = 3-2r\sin\varphi$

$r\cos\varphi + 2r\sin\varphi = 3$

$r = \frac{3}{\cos\varphi+2\sin\varphi}$

(4) $\vec{F} = (y+e^x,\ 3x^2-5y)$ Pres Greenovu větu

C: pozitivně orientovaná $x^2+y^2=y$

$x^2+(y-1/2)^2 = 1/4$

parametrizace kružnice:

$\varphi(t) = (1/4\cos t,\ 1/4\sin t + 1/2)$

$t\in(0,2\pi)$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 6x - 1$$

$$\int_0^{2\pi} \frac{3}{2}\cos t - 1\, dt$$

$$\frac{6}{4}\cos t - 1$$

$$\frac{3}{2}\cos t - 1$$

$$\left[\frac{3}{2}\sin t - t\right]_{t=0}^{t=2\pi} = \underline{\underline{-2\pi}}$$

(5.) Objem tělesa

$x^2+z^2 \le 4$

$z \ge 0$

$y \ge 0$

$x+y+z \le 7$

subst. $z=y$

$y=z$

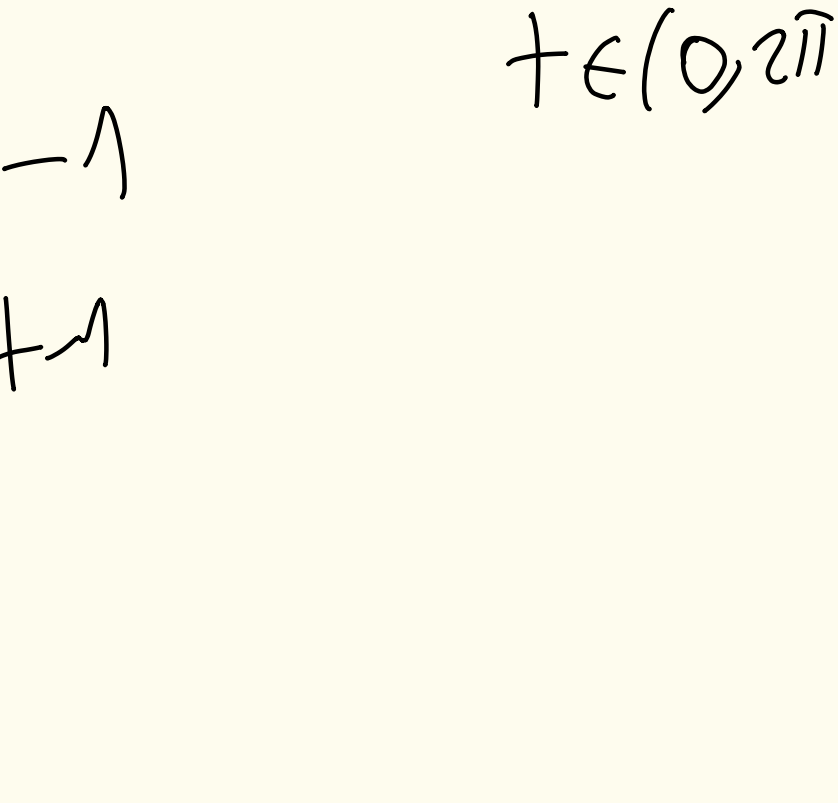
$x=x$

$x^2+y^2\le 4$

$y\ge 0$

$z\ge 0$

$x+y+z\le 7$



V: $0 \le \varphi \le \pi$

$0 \le r \le 2$

$0 \le h \le \frac{7}{r\cos\varphi + r\sin\varphi}$

$$\int_0^{\pi}\int_0^{2}\int_0^{7-r\cos\varphi-r\sin\varphi} r\,dh\,dr\,d\varphi$$

$$r\cos\varphi + r\sin\varphi + h = 7$$

$$h = 7 - r\cos\varphi - r\sin\varphi$$

$$\int_0^{\pi}\int_0^{2} r\left(7-r\cos\varphi-r\sin\varphi\right)\,dr\,d\varphi = \int_0^{\pi}\int_0^{2} 7r - r^2\cos\varphi - r^2\sin\varphi\,dr\,d\varphi =$$

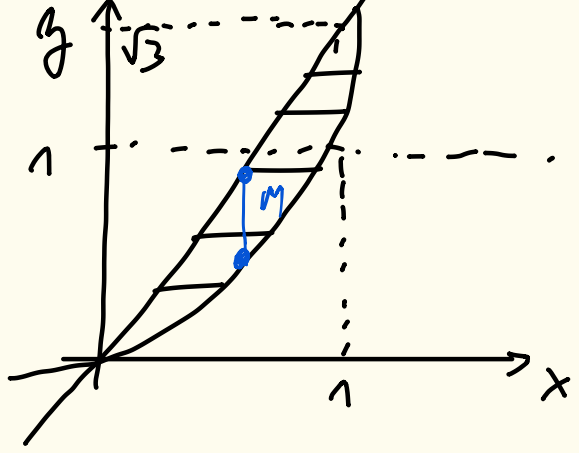
$$= \int_0^{\pi}\left[\frac{7r^2}{2} - \frac{r^3}{3}\cos\varphi - \frac{r^3}{3}\sin\varphi\right]_{r=0}^{r=2} d\varphi =$$

$$= \int_0^{\pi} 14 - \frac{8}{3}\cos\varphi - \frac{8}{3}\sin\varphi\,d\varphi =$$

$$= \left[14\varphi - \frac{8}{3}\sin\varphi + \frac{8}{3}\cos\varphi\right]_{\varphi=0}^{\varphi=\pi} =$$

$$= 14\pi - \frac{8}{3} + \frac{8}{3} = \underline{\underline{14\pi}}$$

$$\int_0^1 \int_{y/\sqrt{3}}^{\sqrt{y}} f\,dx\,dy$$



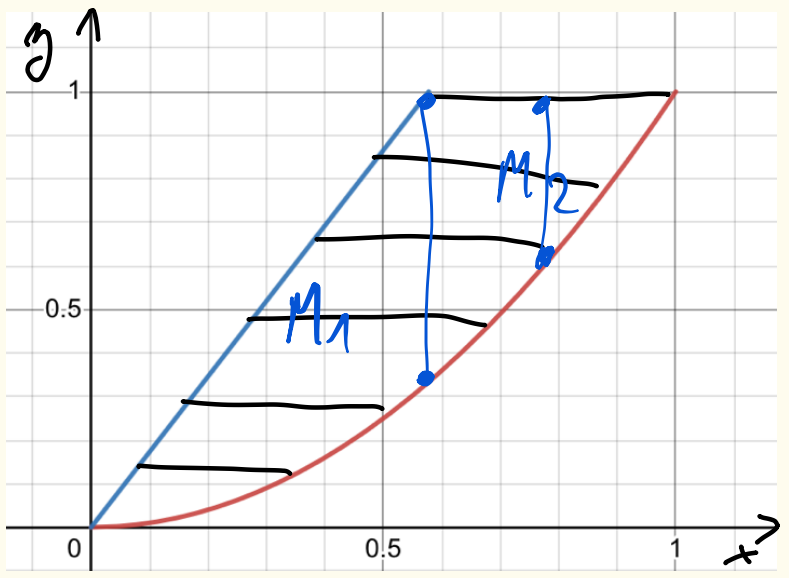
$x = \frac{1}{\sqrt{3}} y$

$x = \sqrt{y}$

$x^2 = y$

$y = \sqrt{3}\,x$

$\approx 1{,}7$



$\sqrt{3}x = 1$

$x = \frac{1}{\sqrt{3}}$

$$\int_0^{1/\sqrt{3}} \int_{\sqrt{3}x}^{x^2} f\,dy\,dx + \int_{1/\sqrt{3}}^{1} \int_{x^2}^{1} f\,dy\,dx$$