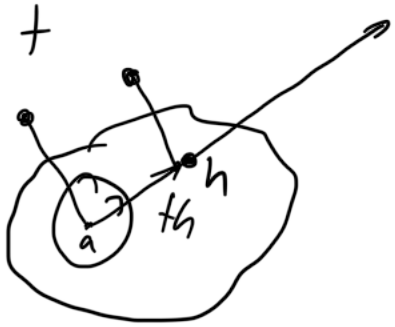
5. Parciální derivace

Definice: Nechť $V \in \mathbb{R}^n$, $f: V \longmapsto \mathbb{R}$, $a \in V$, $h \in \mathbb{R}^n$.

Potom směrová derivace v bodě a, a směru h je

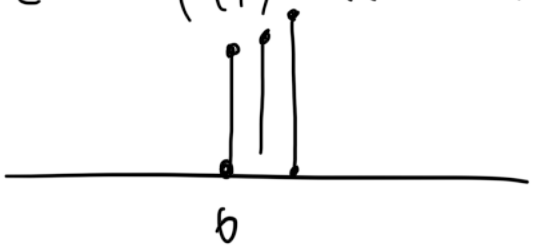
$$\frac{\partial f}{\partial h}(a) = \lim_{t \to 0} \frac{f(a+th) - f(a)}{t}$$



alternativní značení $\partial_h f(a)$

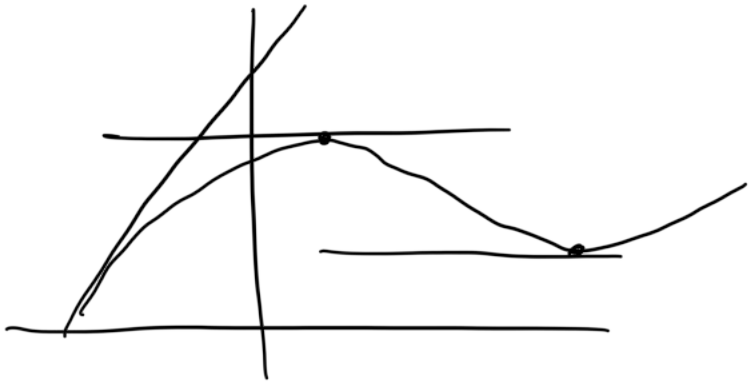
$t \longmapsto a + th$

zkol $\varphi(t) = f(a+th)$



$\varphi(0) = f(a)$

$$\frac{\partial f}{\partial h}(a) = \lim_{t \to 0} \frac{\varphi(t) - \varphi(0)}{t} = \varphi'(0)$$



Tvrzení: $f(a+th) = f(a) + kt + o(t)$ platí

$\Updownarrow$

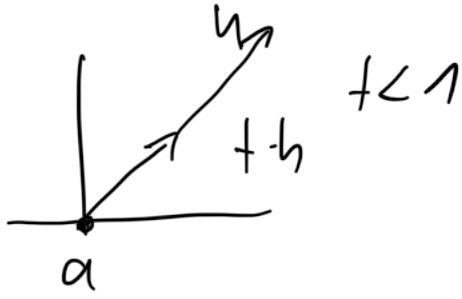
$\frac{\partial f}{\partial h}(a)$ existuje a $= k$

Def: $\varphi : \mathbb{R} \to \mathbb{R}$

$\varphi'(0) = h \Leftrightarrow \varphi(t) = \varphi(0) + \varphi'(0)\cdot t + o(t)$

Tvrzení: Nechť $t \in \mathbb{R}$, $h \in \mathbb{R}^n$, pokud:

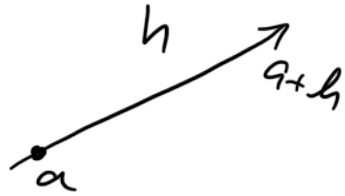
$$\frac{\partial f}{\partial (th)}(a) = t \cdot \frac{\partial f}{\partial h}(a)$$



$$f(a+\gamma\cdot th) = f(a) + \frac{\partial f}{\partial h}(a)\cdot\gamma\cdot t + o(\gamma\cdot t)$$

$$f(a+\gamma\cdot th) = f(a) + \frac{\partial f}{\partial (th)}(a)\cdot\gamma + o(\gamma)$$

Tvrzení Pokud f má $\partial_h f(x)\ \forall x \in a(a+h)$



Pak platí: $f(a+h) - f(a) = \int_0^1 \frac{\partial f}{\partial h}(a+th)\,dt$

Dh: $\varphi(t) = f(a+th)$ φ je diferencovatelná

$\forall t \in (0,1)$. Podle základní věty analýzy

$$\varphi(1) - \varphi(0) = \int_0^1 \varphi'(t)\,dt$$

$\varphi(1) = f(a+h)$

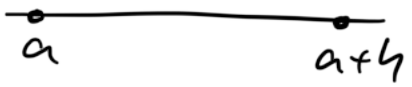
$\varphi(0) = f(a)$

$$\varphi'(t) = \lim_{\gamma \to 0} \frac{\varphi(t+\gamma) - \varphi(t)}{\gamma}$$

$$= \lim_{\gamma \to 0} \frac{f(a+th+\gamma h) - f(a+th)}{\gamma} =$$

$$= \partial_h f(a+th)$$

Tvrzení Nechť f má $\partial_h f(x)$ $\forall x \in a(a+h)$ (spojitá v x) Potom $\exists \gamma \in [0,1]$



$$f(a+h) - f(a) = \frac{\partial f}{\partial h}(a+\gamma h)$$

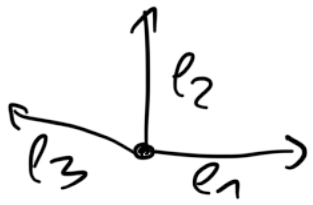
Dh. Zvolím $\varphi(t) = f(a+th)$ (φ' je spojitá $t \in (0,1)$)

$$\varphi(1) - \varphi(0) = \varphi'(\gamma)$$

Definice (Parciální derivace)

$f: U \to \mathbb{R}$, $U \subseteq \mathbb{R}^n$ otevřená

$$\frac{\partial f}{\partial x}(a) := \frac{\partial f}{\partial e_i}(a)$$



$i = 1, \dots, n$

$e_i = (0, \dots, \overset{i}{1}, 0 \dots 0)$

$f(x_1, \dots, x_n) = ?$

Př. $f(x,y) = x^2 + xy^3 + y$

$\frac{\partial f}{\partial x}$ $\frac{\partial f}{\partial y}$

$f(x,y) = x^2 + xy^3 + y$ $\quad \frac{\partial f(x,y)}{\partial e_1} = \frac{\partial f}{\partial x}(x,y)$

$\varphi(t) = f((x,y) + t \cdot (1,0))$

$\varphi(t) = f(x+t, y)$

$\varphi'(0) = \frac{\partial}{\partial x} f(x,y)$

$f(x+t, y) = (x+t)^2 + (x+t)y^3 + y$

$\varphi(t)$

$\varphi'(0) = 2(x+t) + 1 \cdot y^3 + 0$ ($t \to 0$)

$= 2x + y^3 = \frac{\partial f}{\partial x}(x,y)$

Definice: Nechť $U \in \mathbb{R}^n$ otevřená, $f: U \to \mathbb{R}$

Pokud $\frac{\partial f}{\partial x_i}(a)$ existují pro $i = 1, \dots, n$. Potom

výraz: $\operatorname{grad} f(a) = \left(\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_n}(a) \right)$

se nazývá gradient. Jde o vektor z $\mathbb{R}^n$.

Př. $f(x,y,z) = x \cdot y^2 \cdot z^4$

$$\operatorname{grad} f(x,y,z) = (y^2 z^4, 2xyz^4, 4xy^2 z^3)$$

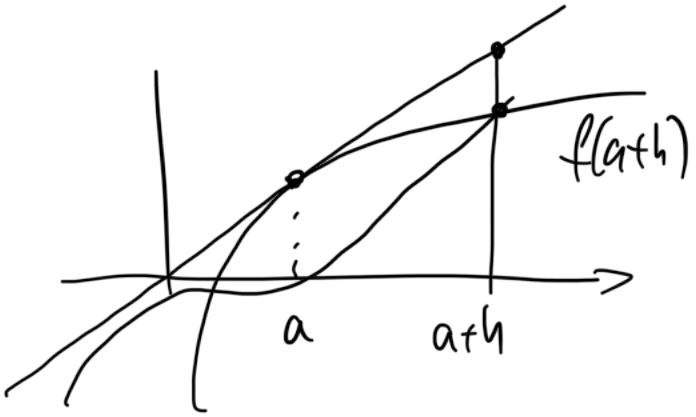
Definice. Nechť $U \subset \mathbb{R}^n$ otevřená, $f: U \to \mathbb{R}$, Derivace (ve starší terminologii diferenciál, totální dif.) v bodě $a \in U$ lineární zobrazení $L: \mathbb{R}^n \to \mathbb{R}$, takové, že platí:

$$\lim_{\|h\| \to 0} \frac{f(a+h) - f(a) - L(h)}{\|h\|} = 0$$

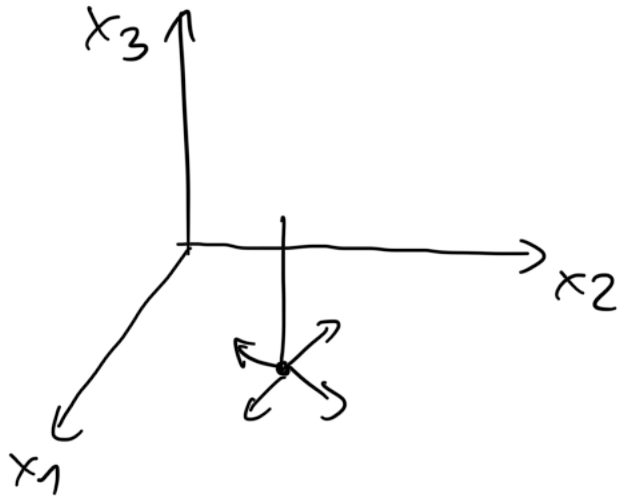
Značení $df(a) = L$.

$$df(a)[h] = L(h)$$

$$\frac{f(a+h) - f(a) - f'(c) \cdot h}{h} \to 0 \qquad h \to 0$$



$$f(a+h) = f(a) + f'(a)h + o(h)$$



$x_3 = f(x_1, x_2)$

$$f(a+h) = \underbrace{f(a) + L(h)}_{} + o(\|h\|)$$

$$f(a+h) - f(a) - L(h) = o(\|h\|)$$

Poznámka Víme, že $\forall$ lineární zobrazení

$L: \mathbb{R}^n \to \mathbb{R}$ $\exists$ maticové reprezentace

$h = (h_1, \dots, h_n) \to Lh =$

$$= \sum_{i=1}^{n} a_i h_i = (a_1, \dots, a_n) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$$

Věta Nechť $U \subset \mathbb{R}^n$ otevřená, $f: U \to \mathbb{R}$. Pak má f v a směrové derivace a platí

$h \in \mathbb{R}^n$

$$df(a)[h] = \operatorname{grad} f(a) \cdot h$$
$$= \operatorname{grad} f(a) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i$$

Tedy platí:

$$f(a+h) = f(a) + \operatorname{grad} f(a) \cdot h + o(\|h\|)$$

Dk: Platí:

$$f(a+h) = f(a) + L\,h + o(\|h\|)$$

Zvolím $h = e_i$ $\quad L = (l_1, \dots, l_n) \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} e_i$

$$f(a + t e_i) = f(a) + l_i \cdot t + o(t)$$

$$\Downarrow$$

$\frac{\partial f}{\partial x_i}$ existuje, $= l_i \Rightarrow L = \left(\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_n} \right)$

Tedy $L = \operatorname{grad} f$