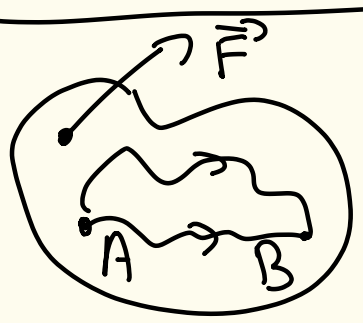
Potenciál, konzervativní pole



$V \subseteq \mathbb{R}^n$ otevřená

$\vec{F}$ vcht. pole

- integrál $\int \vec{F} \, d\vec{s}$ nezávisí na cestě ale jen na začátku a konci

$\oint \vec{F} \, d\vec{s} = 0$ přes každou uzavřenou křivku je nulový

$\exists f: V \to \mathbb{R}$, že $\forall a \in V \; grad(f)(a) = \vec{F}(a)$

(ekvivalentní)

potenciál pro $\vec{F}$

$\vec{F}$ je konzerv. pole

$\vec{F}$ je hladké

$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \; \forall i,j$

pro $n=3$ odpovídá $rot(\vec{F}) = \vec{0}$

vzájemně inverzní operace

$$\int_a^b \vec{F} \, d\vec{A} = f(B) - f(A)$$

Určete hodnotu parametru $\alpha \in \mathbb{R}$ tak, aby pole $\vec{F}(x,y,z) =$

$= (y+z, x+z, x+\alpha y)$

bylo konzervativní a najděte potenciál.

V případě konzervativnosti určete práci pole $\vec{F}$ z bodu $A=(0,0,0)$

do bodu $B=(1,1,1)$

$D(\vec{F})=\mathbb{R}^3$

↑ jednoduše souvislá množina

$\vec{F}$ je konzervativní $\Leftrightarrow rot(\vec{F})=\vec{0}$

⋮

potenciál pro α

vše vyřešíme společným postupem (až na $\int_A^B \dots$)

hledáme fci $f: \mathbb{R}^3 \to \mathbb{R}$, že

$\frac{\partial f}{\partial x}=y+z \Leftrightarrow f(x,y,z)=\int \frac{\partial f}{\partial x}\,dx$

$=yx+zx+C(y,z)$

(C má jako integrační konstanta, protože $C(y,z)=f(x,y,z)-(yx+zx)$)

$\frac{\partial f}{\partial y}=x+z \Leftrightarrow xz=\frac{\partial}{\partial y}\left|yx+zx+C(y,z)\right|$ dosadíme

$\frac{\partial f}{\partial z}=x+\alpha y \Leftrightarrow \| = x+\frac{\partial C}{\partial y}(y,z)$

$\frac{\partial C}{\partial y}(y,z)=z \Leftrightarrow C(y,z)=\int z\,dy=zy+D(z)$

dosadíme

$\Rightarrow yx+zx+zy+D(z)$

$\Rightarrow x+\alpha y=\frac{\partial}{\partial z}\left(yx+zx+zy+D(z)\right)$

$x+y+\frac{\partial D}{\partial z}(z)$

$\frac{\partial D}{\partial z}(z)=(\alpha-1)y$

pro $\alpha \neq 1$ rovnice nemá řešení (potenciál neexistuje)

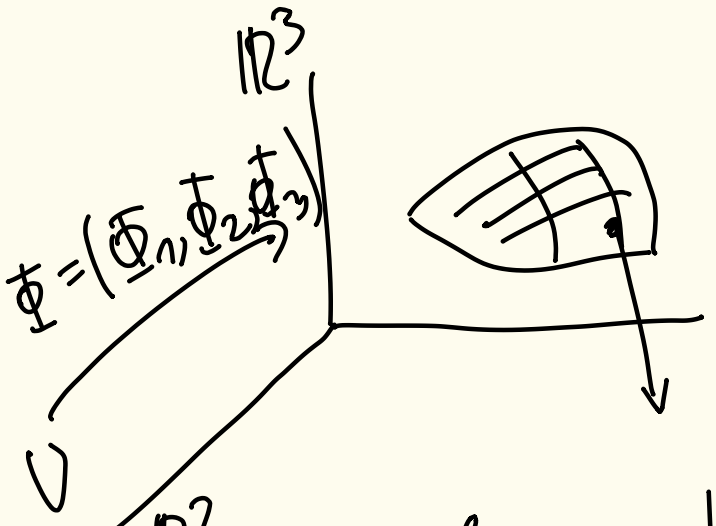
$$\frac{\partial D}{\partial z}(z) = 0 \iff D = \text{konst} \dots$$

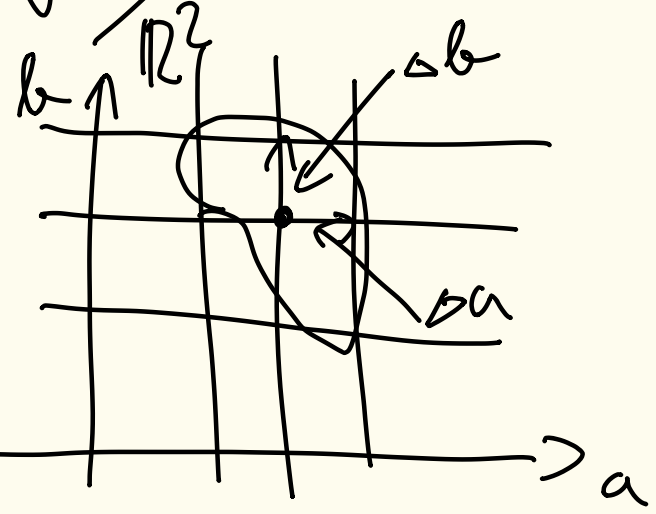
potenciál $f(x,y,z) = yx + zx + zy$

$$\int_A^B \vec{F}\, d\vec{s} = f(B) - f(A) = f(1,1,1) - f(0,0,0)$$

$$= 3 - 0 = 3$$

Plošný integrál





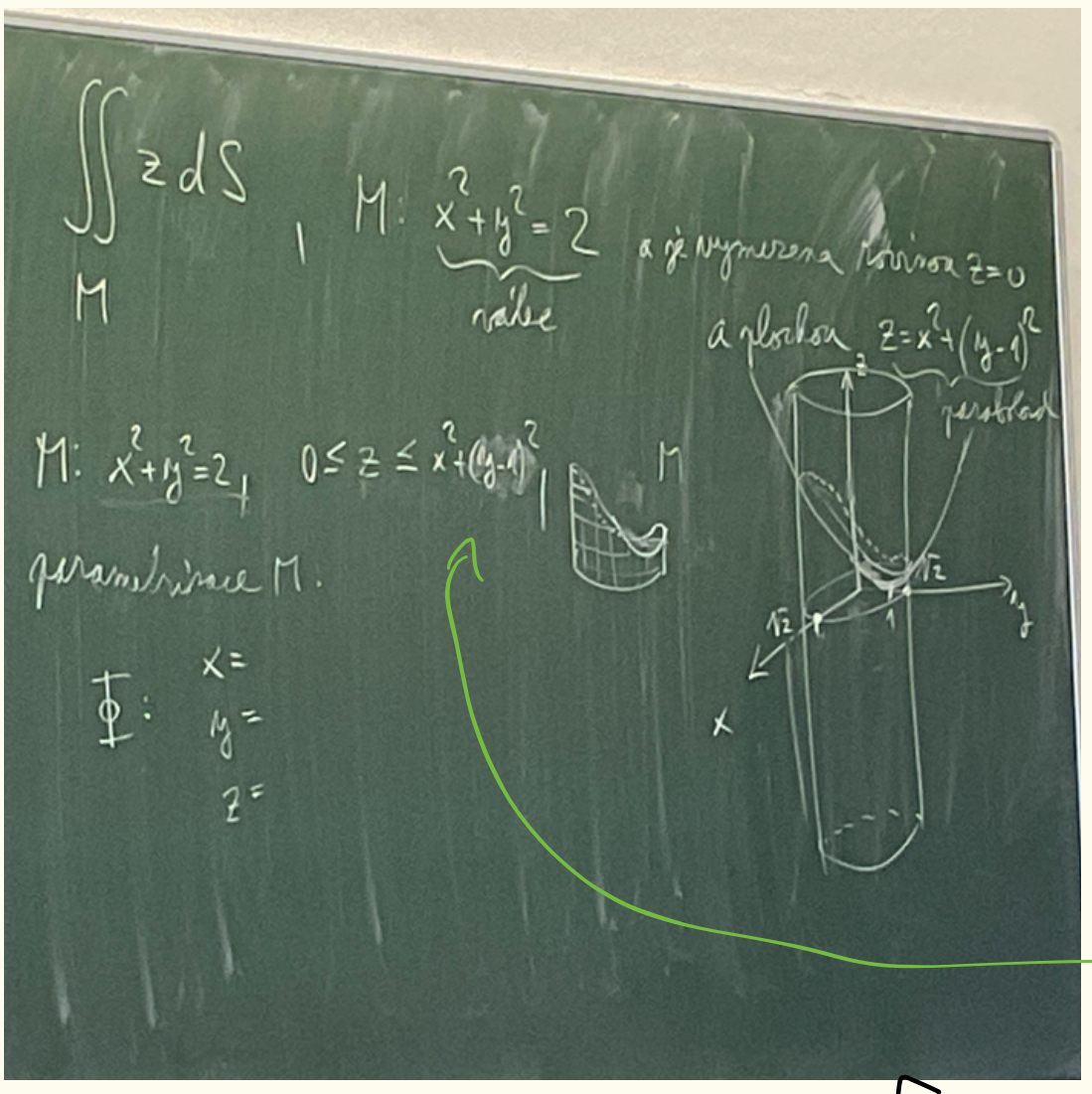
$(b| \cdot \Delta a$

$$d\Phi(a,b) : \mathbb{R}^2 \to \mathbb{R}^3 = \frac{\partial \Phi}{\partial b}$$

$$\frac{\partial \Phi}{\partial a} = \begin{pmatrix} \frac{\partial \Phi_1}{\partial a} & \frac{\partial \Phi_1}{\partial b} \\ \vdots & \vdots \\ \frac{\partial \Phi_3}{\partial a} & \frac{\partial \Phi_3}{\partial b} \end{pmatrix}$$

$$\iint_M f\, dS = \iint_V f(\Phi(a,b)) \left\| \frac{\partial \Phi}{\partial a} \times \frac{\partial \Phi}{\partial b} \right\| da\, db$$

$$\iint_M z\,dS, \quad M: \underbrace{x^2+y^2=2}_{\text{válec}} \text{ a je vymezena rovinou } z=0 \text{ a plochou } z=x^2+(y-1)^2$$



dosadíme

parametrizace M: $\Phi: \begin{cases} x=\sqrt{2}\cos\varphi \\ y=\sqrt{2}\sin\varphi \\ z=h \end{cases}$

$$2=2$$

$$0\le h\le 2\cos^2\varphi+2\sin^2\varphi-2\sqrt{2}\sin\varphi+1=$$

$$=3-2\sqrt{2}\sin\varphi$$

$$0\le 3-2\sqrt{2}\sin\varphi \iff \sin\varphi\le\frac{3}{2\sqrt{2}} \quad \underbrace{\frac{3}{4}\sqrt{2}}_{\vee}$$

$$\exists h: \quad 4>1 \ \checkmark$$

$$\frac{\partial\Phi}{\partial\varphi}\left(\frac{\partial x}{\partial\varphi},\frac{\partial y}{\partial\varphi},\frac{\partial z}{\partial\varphi}\right)=(-\sqrt{2}\sin\varphi,\sqrt{2}\cos\varphi,0)$$

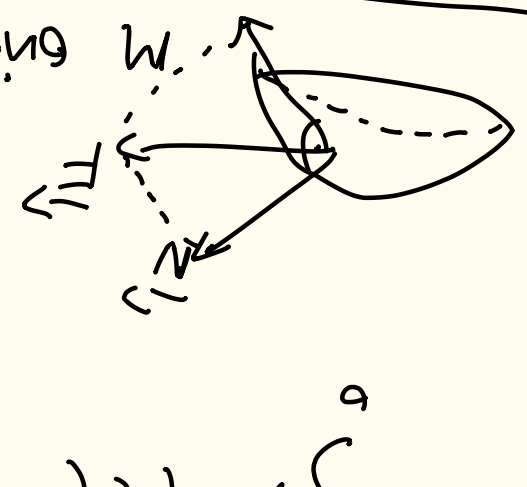
$$\frac{\partial\Phi}{\partial h}=(0,0,1)$$

vektory jsou lineární

$$\left\|\frac{\partial\Phi}{\partial\varphi}\times\frac{\partial\Phi}{\partial h}\right\|=\left\|\frac{\partial\Phi}{\partial\varphi}\right\|\cdot\left\|\frac{\partial\Phi}{\partial h}\right\|$$

$$\iint_M \vec{E}\,d\vec{S} := \iint_M (\vec{E}\cdot\vec{N})\,dS = \iint_U \vec{E}(\vec{\Phi}(\xi,\varphi))\cdot\left(\frac{\partial\vec{\Phi}}{\partial\eta}\times\frac{\partial\vec{\Phi}}{\partial\varphi}\right)d\xi$$

$\vec{N}$ ... Einheitsnormalenvektor (positiv/nach außen gerichtet)



$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} q - A2\sqrt{2}\sin\varphi + 8\cos^2\varphi\,d\varphi = \frac{\sqrt{2}}{2}\left(q\cdot 2\pi + 8\pi\right) =$$

$$= 4\pi\sqrt{2}\,\Pi$$

$$\int_{\pi}^{2\pi} 9\cos^2\varphi\,d\varphi = \Pi$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left[\frac{h^2}{2}\right]_0^{3-2\sqrt{2}\sin\varphi} d\varphi = \frac{\sqrt{2}}{2}\int_0^{2\pi}\left(3-2\sqrt{2}\sin\varphi\right)^2 d\varphi =$$

$$\iint_M z\,dS = \iint_U h\cdot\left\|\frac{\partial\vec{\Phi}}{\partial h}\times\frac{\partial\vec{\Phi}}{\partial\varphi}\right\| dh\,d\varphi = \int_0^{2\pi}\int_0^{3\sqrt{2}\sin\varphi} \sqrt{2}\,h\,dh\,d\varphi =$$

$$\frac{\partial\vec{\Phi}}{\partial\varphi}\times\frac{\partial\vec{\Phi}}{\partial h} = \begin{pmatrix}\sqrt{2}\cos\varphi\\ \sqrt{2}\sin\varphi\\ 0\end{pmatrix}$$

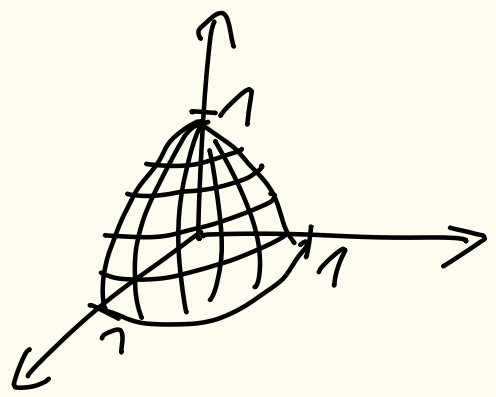
$\|\sqrt{2}\|$

+ hdyž je vektor na opačnou stranu jakoN
- naopak...

$$\iint_M \vec{F}\,d\vec{S}, \quad M: x^2+y^2+z^2=1 \quad y\geq 0,\ z\geq 0,\ x\geq 0$$

orientace M je směrem do poloprostoru (dolů)

$$\vec{F}=(x,-y,z)$$



3. souřadnice pole je zápornou

parametrizace M — sférické souřadnice — jako graf funkce ← zhusíme toto

$$z=\sqrt{1-x^2-y^2}$$

$$\Phi(x,y)=(x,y,\sqrt{1-x^2-y^2})$$

$$U: x^2+y^2\leq 1,\ x,y\geq 0$$

$$\frac{\partial\Phi}{\partial x}=\left(1,0,-\frac{x}{\sqrt{1-x^2-y^2}}\right)$$

$$\frac{\partial\Phi}{\partial y}=\left(0,1,-\frac{y}{\sqrt{1-x^2-y^2}}\right)$$

$$\frac{\partial\Phi}{\partial x}\times\frac{\partial\Phi}{\partial y}=\left(\frac{x}{\sqrt{1-x^2-y^2}},\frac{y}{\sqrt{1-x^2-y^2}},1\right)$$

$$\iint_M \vec{F}\,d\vec{S} = -\iint_U \vec{F}\left(\vec{\Phi}(x,y)\right)\cdot\left(\frac{\partial\vec{\Phi}}{\partial x}\times\frac{\partial\vec{\Phi}}{\partial y}\right)dx\,dy =$$

$$= -\iint_U \left(x,\ -\sqrt{1-x^2-y^2},\ y\right)\cdot\begin{pmatrix}\frac{x}{\sqrt{1-x^2-y^2}}\\ \frac{y}{\sqrt{1-x^2-y^2}}\\ 1\end{pmatrix}dx\,dy =$$

$$= -\iint \frac{x^2}{\sqrt{1-x^2-y^2}} - y + y\,dx\,dy =$$

$$= \left\{\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \\ 0\le r\le 1 \\ 0\le\varphi\le\frac{\pi}{2}\end{array}\right\} = \int_0^{\pi/2}\int_0^1 \frac{r^2\cos^2\varphi}{\sqrt{1-r^2}}\cdot r\,dr\,d\varphi =$$

$$= \left(-\underbrace{\int_0^{\pi/2}\cos^2\varphi\,d\varphi}_{-\frac{\pi}{4}}\right)\cdot\left(\int_0^1 \frac{r^3}{\sqrt{1-r^2}}\,dr\right) = -\frac{\pi}{6}$$

$$\left\{\begin{array}{l} s = 1-r^2 \\ ds = -2r\,dr\end{array}\right\} = \frac{1}{2}\int_1^0 \frac{1-s}{\sqrt{s}}\,ds =$$

$$= \frac{1}{2}\int_0^1 s^{-\frac{1}{2}} - s^{\frac{1}{2}}\,ds = \left[s^{\frac{1}{2}}\cdot\frac{4}{3}s^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}$$