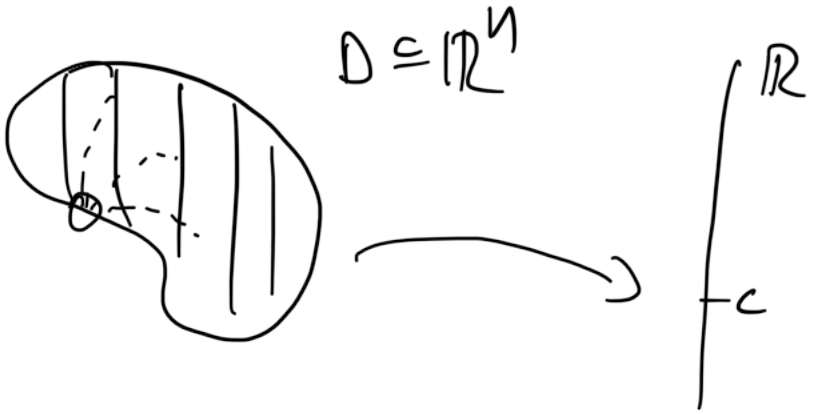
Limita



- a_0 je hromadný bod množiny D

$$(\forall \varepsilon > 0 \; P_\varepsilon(a_0) \cap D \neq \emptyset)$$

- $c \in \mathbb{R}$ je limita funkce f v bodě a_0 (znač. $\lim\limits_{a \to a_0} f(a) = c$)

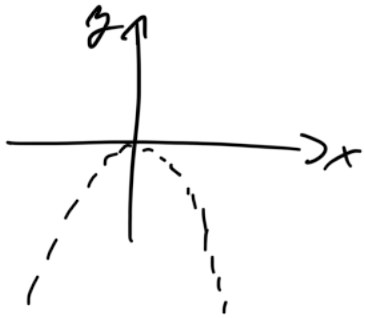
$$\Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall a \in D \; a \in P_\delta(a_0) \Rightarrow f(a) \in U_\varepsilon(c)$$

$$0 < \|a - a_0\| < \delta \Rightarrow |f(a) - c| < \varepsilon$$

Vyšetřete existenci limity

$$\lim_{(x,y) \to (0,0)} \underbrace{\frac{y^2 - x}{x^2 + y}}_{\| \; f(x,y)}$$

$D(f): x^2 = -y$



zkusíme přiblížení po parabole $x = y^2$ $\quad f(y^2, y) = 0 \xrightarrow{y \to 0} 0$

zkusíme přímku $y = kx$

$$\lim_{x \to 0} \frac{k^2 x^2 - x}{x^2 + kx} = \lim_{x \to 0} \frac{x(k^2 x - 1)}{x(x + k)} = \lim_{x \to 0} \frac{k^2 x - 1}{x + k} =$$

$$\frac{\lim_{x\to 0} \overrightarrow{k^2 x} - \lim_{x\to 0} 1}{\lim_{x\to 0} x + \lim_{x\to 0} k} = -\frac{1}{k} \neq 0$$

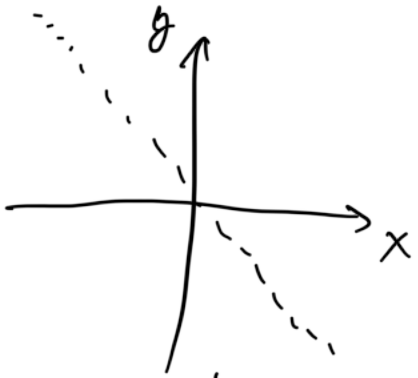
$\lim_{x\to 0} x \to 0$

$\lim_{x\to 0} k \to 0$

↳ různé limity pro různá k, limita neexistuje

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x+y}$$

$D(f): x \neq -y$



$y = kx$

$$\lim_{x\to 0} \frac{xkx}{x+kx} = \frac{x^2 k}{x(k+1)} = \lim_{x\to 0} \frac{xk}{k+1} = 0$$

$\lim_{t\to 0}$

přímka $x = 0$

$$f(0,y) = \frac{0y}{0+y} = 0 \xrightarrow{y\to 0} 0$$

$y = k(x)\,x$ $\quad k(x) = x - 1$

$$\frac{x^2 \cdot h(x)}{x(1+k(x))} = \frac{x\,h(x)}{1+h(x)} \to -1$$

kritérium neexistence (konečné) limity

limita typu $\lim_{a\to a_0} \frac{g(a)}{h(a)}$

- funkce g a h jsou spojité na nějakém ryzím okolí bodu a_0

- existuje množina M takové, že

1) $\forall a \in M \quad g(a) \neq 0 \wedge$
$h(a) = 0$

2) a_0 je hromadný bod množiny M

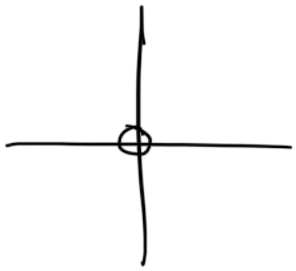
Pak neexistuje konečná limita
$$\lim_{a \to a_0} \frac{g(a)}{h(a)}$$

$g(x,y) = x \cdot y$ $\quad M: x = -y, x \neq 0$
$h(x,y) = x + y$

limita neexistuje

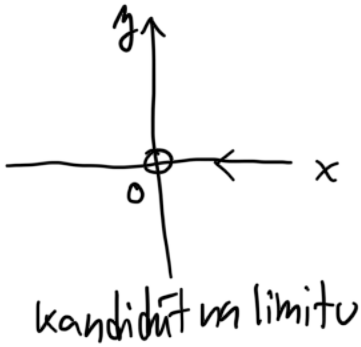
$$\lim_{(x,y)\to(1,0)} \frac{(x-1)^4 + y^4}{(x-1)^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{x^4 + y^4}{x^2 + y^2}$$

$D(f) = \mathbb{R}^2 \setminus \{(0,0)\}$



přiblížení po ose x:

$$f(x,0) = \frac{x^4}{x^2} = x^2 \xrightarrow{x\to 0} 0$$
$$\| \quad c$$



kandidát na limitu

asi limita existuje
použijeme odhad: (a věta o dvou policajtech)

$$\lim_{a\to a_0} f(a) = c \iff \lim_{a\to a_0} |f(a) - c| = 0$$

$$0 \le |f(x,y) - c| = \frac{x^4+y^4}{x^2+y^2} = \frac{|x|^4 + |y|^4}{\left(\sqrt{x^2+y^2}\right)^2} \le \frac{2\left(\sqrt{x^2+y^2}\right)^4}{\left(\sqrt{x^2+y^2}\right)^2} =$$

$$x^2 \le x^2+y^2$$

$$|x| \le \sqrt{x^2+y^2}$$

$$|y| \le$$

$$= 2\left(\sqrt{x^2+y^2}\right)^2 = 2(x^2+y^2)$$

↓ 2. policajt

↓ $(x,y) \to (0,0)$

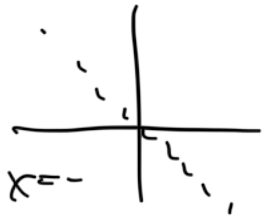
$$0$$

$$\frac{x^4+y^4}{x^2+y^2} \le 2\sqrt{x^2+y^2}$$

$$\Downarrow$$

limita je $c=0$

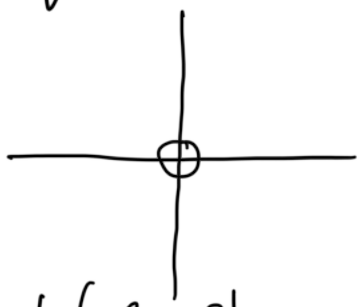
$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^3+y^3}$$ nee



$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{|x|^3+|y|^3} \quad \dots = \frac{|x|^4+|y|^4}{\left(\sqrt[3]{|x|^3+|y|^3}\right)^3} \le \quad \dots$$

$$\lim_{(x,y)\to(0,0)} \underbrace{(x^2+y^2)^{x^2y^2}}_{f(x,y)} = e^{x^2y^2\ln(x^2+y^2)}$$

$D(f) = \mathbb{R}^2 \setminus \{(0,0)\}$

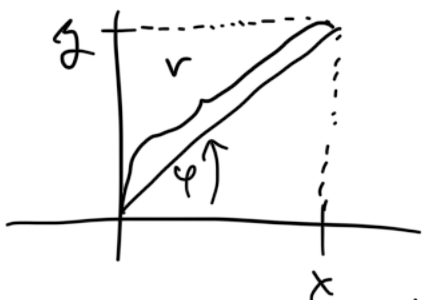


chceme zjistit $\lim_{(x,y)\to(0,0)} x^2y^2 + \ln(x^2+y^2)$

přiblížení pro ose y dá vrhodnotu $0 = c$

kandidát

zkusíme použít polární souřadnice



$x = r\cos\varphi$

$y = r\sin\varphi$, $\varphi \in \langle 0, 2\pi)$

$r \neq 0$ (pak je to popis bodů z $D(f)$)

$$g(r\cos\varphi, r\sin\varphi) = r^4\cos^2\varphi\,\sin^2\varphi\,\ln r^2$$

ptáme se, jaká je limita

$$\lim_{r\to 0+} r^4 \underbrace{\ln(r^2)}_{= 2\ln r} = \lim_{r\to 0+} \frac{2\ln(r)}{\frac{1}{r^4}} \overset{l'H}{=} \lim_{r\to 0+} \frac{\frac{2}{r}}{\frac{-4}{r^5}} =$$

$$= \lim_{r\to 0^+} -\frac{1}{2}r^4 = 0$$

$$\left| f(r\cos\varphi, r\sin\varphi) - c \right| = r^4 \ln(r^2) \cdot \underbrace{\cos^2\varphi}_{\leq 1} \underbrace{\sin^2\varphi}_{\leq 1} \leq r^4 \ln(r^2) =$$

$$r = \sqrt{x^2+y^2} \qquad = (x^2+y^2)^2 \cdot \ln(x^2+y^2)$$

víme: $h(r) = r^4 \ln r^2$

1) $\lim\limits_{r \to 0+} h(r) = 0$

2) $s(x,y) = \sqrt{x^2+y^2}$

$\lim\limits_{(x,y)\to(0,0)} s(x,y) = 0$

3) $(x^2+y^2)^2 \cdot \ln(x^2+y^2) = h(\underbrace{s(x,y)}_{\to 0}) \to 0$ věta o limitě složené fce

věta o policajtech

$\Longrightarrow \lim\limits_{(x,y)\to(0,0)} g(x,y) = 0 \Rightarrow \lim\limits_{(x,y)\to(0,0)} e^{g(x,y)} = \boxed{1}$