Nezávislost náhodných veličin X a Y:

(Náh. jevy jsou nezávislé, jestliže $P(A \cap B) = P(A)P(B)$)

$$P(X \leq i \cap Y \leq j) =$$

$$= P(X \leq i) \cdot P(Y \leq j) \quad \forall i,j \in \mathbb{R}$$

Máme nezávislé n.v. X a Y $\longrightarrow$ zajímá nás rozdělení $Z = X + Y$

$\downarrow$

konvoluce n.v.

Př.: Hra: házíme kostkou a mincí. Výhry jsou:

kostka: 1, 2 nebo 3 $\longrightarrow$ 1 EUR

4 nebo 5 $\longrightarrow$ 2 EUR

6 $\longrightarrow$ 3 EUR

mince: panna $\rightarrow$ 1 EUR

orel $\rightarrow$ 2 EUR

X ... výhra na kostce

Y ... výhra na minci

Celková výhra $Z = X + Y \rightarrow$ Rozdělení Z?

$$P(Z=2) = P(X=1 \cap Y=1) = P(X=1)\,P(Y=1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P(Z=3) = P(X=1) \cdot P(Y=2) + P(X=2) \cdot P(Y=1) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{3+2}{12} = \frac{5}{12}$$

$$P(Z=4) = P(X=1) \cdot P(Y=3) + P(X=2)\,P(Y=2) + P(X=3)\,P(Y=1) = \frac{1}{2} \cdot 0 + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{2} = \frac{3}{12} = \frac{1}{4}$$

$$P(Z=5) = P(X=1)\,P(Y=4) + \ldots + P(X=3)\,P(Y=2) = \frac{1}{2} \cdot 0 + \frac{1}{3} \cdot 0 + \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$$

Diskrétní X a $Y \longrightarrow Z = X + Y$. $P(Z=k) = \sum_{i=0}^{k} P(X=i)\,P(Y=k-i)$

Spojité X a $Y \longrightarrow Z = X + Y$: $h(z) = \int_{-\infty}^{\infty} f(x)\,g(z-x)\,dx$

hustoty $\quad h \quad f \quad g$

1) $X \sim \boxed{Alt(p)}$, tj. $P(X=1)=p$, $P(X=0)=1-p$
$Y \sim \boxed{-\,''\,}$ $P(Y=1) = -\,''\,-$ $P(Y=0) = -\,''\,-$
$\Leftrightarrow Binom(1,p)$

$P(Z=0) = (1-p)^2 = \binom{2}{0} p^0 (1-p)^2$
$P(Z=1) = p(1-p) + (1-p)\cdot p = 2p(1-p) = \binom{2}{1} p^1 (1-p)^1$
$P(Z=2) = p^2 = \binom{2}{2} p^2 (1-p)^0$
$\Rightarrow Z = X+Y \sim Binom(2,p)$

2) $X \sim Binom(n_1,p)$, tj. $P(X=i) = \binom{n_1}{i} p^i (1-p)^{n_1-i}$
$Y \sim Binom(n_2,p)$, tj. $P(Y=j) = \binom{n_2}{j} p^j (1-p)^{n_2-j}$
$Z = X+Y$, ozn $n_1+n_2 = n$

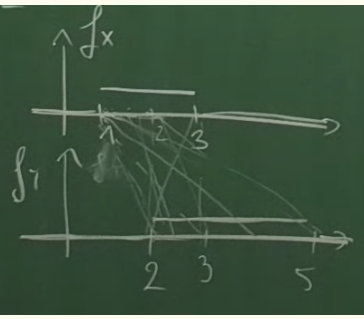
$$P(Z=k) = \sum_{i=0}^{k} P(X=i)\,P(Y=k-i) = \sum_{i=0}^{k} \binom{n_1}{i} p^i (1-p)^{n_1-i} \binom{n_2}{k-i} p^{k-i} (1-p)^{n_2-k+i} =$$
$$= p^k (1-p)^{\overbrace{n_1+n_2}^{n}-k} \cdot \underbrace{\sum_{i=0}^{k} \binom{n_1}{i}\binom{n_2}{k-i}}_{\overset{?}{=}\binom{n}{k}} = \binom{n}{k} p^k (1-p)^{n-k} \Rightarrow Z \sim Binom(\underset{n_1+n_2}{n}, p)$$

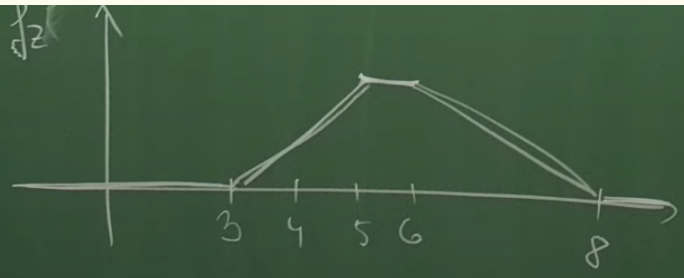
Binom věta $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$
$$(x+1)^{n_1}(x+1)^{n_2} = \sum_{i=0}^{n_1} \binom{n_1}{i} x^i \sum_{j=0}^{n_2} \binom{n_2}{j} x^j \Rightarrow j = k-i$$
pro pevné k: $\binom{n}{k} = \sum_{i=0}^{k} \binom{n_1}{i}\binom{n_2}{k-i}$

3.) $X \sim Po(\lambda_1)$
$Y \sim Po(\lambda_2)$
$Z = X+Y$
ozn $\lambda = \lambda_1 + \lambda_2$

$$P(Z=k) = \sum_{i=0}^{k} P(X=i)\cdot P(Y=k-i) = \sum_{i=0}^{k} \frac{\lambda_1^i}{i!} e^{-\lambda} \cdot \frac{\lambda_2^{k-i}}{(k-i)!} e^{-\lambda_2} \cdot \frac{k!}{k!} = \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \cdot \underbrace{\sum_{i=0}^{k} \underbrace{\frac{k!}{i!(k-i)!}}_{\binom{k}{i}} \cdot \lambda_1^i \lambda_2^{k-i}}_{\text{Binom věta } (\lambda_1+\lambda_2)^k} = \frac{\lambda^k}{k!} e^{-\lambda} \Rightarrow$$
$\Rightarrow Z \sim Po(\lambda)$

4) $X \sim Ro(1,3)$
$Y \sim Ro(2,5)$
$Z = X+Y$





$X_1 \sim Exp(\lambda)$, tj. $f(x) = \lambda e^{-\lambda x} \Rightarrow$ Rozdělení $X_1 + \dots + X_k \to f_k(x)$
hustota
$k \geq 1$

pro $i = 1, 2, \dots \infty$

$$f_2(x) = \int_{-\infty}^{\infty} f(t) f(x-t)\,dt = \int_0^x \lambda e^{-\lambda t} \lambda e^{-\lambda(x-t)}\,dt =$$
$$= \lambda^2 e^{-\lambda x} \int_0^x 1\,dt = \lambda^2 e^{-\lambda x} [t]_0^x = \lambda^2 e^{-\lambda x} x$$

pro $x > 0$
$(f_2(x) = 0$ pro $x \leq 0)$

$$f_3(x) = \int_0^x f_2(t) f(x-t)\,dt = \int_0^x \lambda^2 e^{-\lambda t} t \lambda e^{-\lambda(x-t)}\,dt =$$
$$= \lambda^3 e^{-\lambda x} \int_0^x t\,dt = \lambda^3 e^{-\lambda x} \left[\frac{t^2}{2}\right]_0^x = \lambda^3 e^{-\lambda x} \frac{x^2}{2}$$

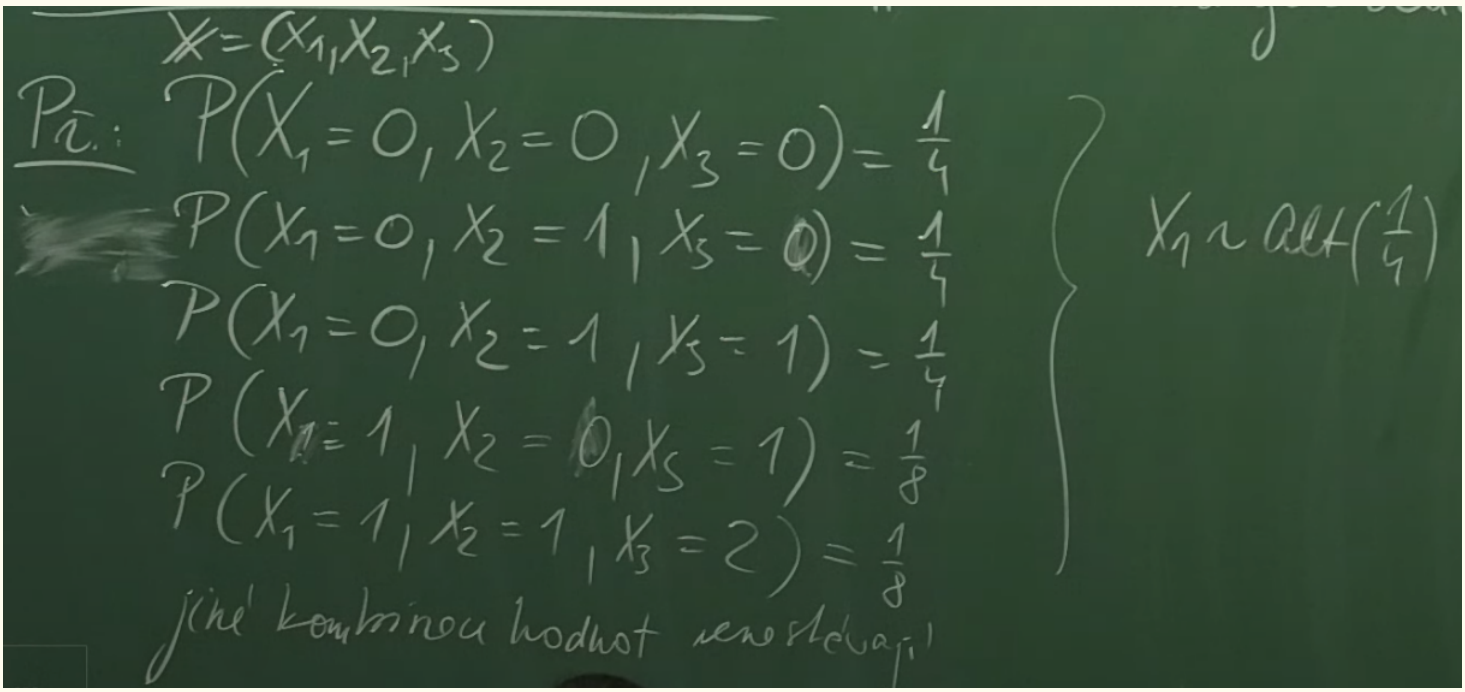
předp $f_k(x) = \frac{\lambda^k}{(k-1)!} e^{-\lambda x} x^{k-1} \overset{?}{\Longrightarrow}$ chceme $f_{k+1}(x) = \frac{\lambda^{k+1}}{k!} e^{-\lambda x} x^k$

$$f_{k+1}(x) = \int_0^x f_k(t) f(x-t)\,dt = \int_0^x \frac{\lambda^k}{(k-1)!} e^{-\lambda t} t^{k-1} \lambda e^{-\lambda(x-t)}\,dt =$$

$$= \frac{\lambda^{k+1}}{(k-1)!} e^{-\lambda x} \left[t^{k-1}\right]_0^x = \frac{\lambda^{k+1}}{(k-1)!} e^{-\lambda x} \cdot \frac{x^k}{k} = \frac{\lambda^{k+1}}{k!} e^{-\lambda x} x^k$$

NÁHODNÝ VEKTOR = „vektor náhodných veličin", tj.

$\mathbb{X} = (X_1, \dots, X_n) \rightarrow$

→ 1. pojetí: n znaků na jednom objektu → zkoumáme souvislosti

→ 2. pojetí: 1 znak zkoumaný na n objektech → statistika