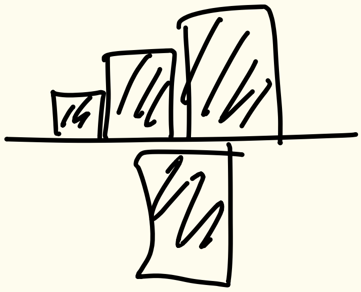
$(3{,}5 =)\ \mathbb{E}X$



$$\mathbb{E}(X-\mathbb{E}X) = \overset{a}{\mathbb{E}X} - \mathbb{E}(\overset{a}{\mathbb{E}X}) = a - a = 0$$

$$\mathbb{E}(X-\mathbb{E}X)^2 = \operatorname{var}X \ (= DX) \longrightarrow \operatorname{sd} X = \sqrt{\operatorname{var}X}$$

směrodatná odchylka

Př. X ... vzdělání ($X=1$ pro ZŠ, $X=2$ pro SŠ, $X=3$ pro VŠ)
Y ... platová třída ($Y=1$ pro nejnižší, ..., $Y=4$ pro nejvyšší)

$\to \mathbb{E}X = 2{,}1$ (např.)

$\to \mathbb{E}Y = 2{,}7$ (např.)

$$\mathbb{E}(X-\mathbb{E}X)(Y-\mathbb{E}Y) > 0 \quad = \quad \operatorname{cov}(X,Y)$$

kovariance

VŠ	>0	>0	$>$
ZŠ	<0	<0	$>$

pokud ale ZŠ = 3 SŠ = 2 VŠ = 1

$$\operatorname{cov}(X,Y) = (X-\mathbb{E}X)(Y-\mathbb{E}Y) < 0$$

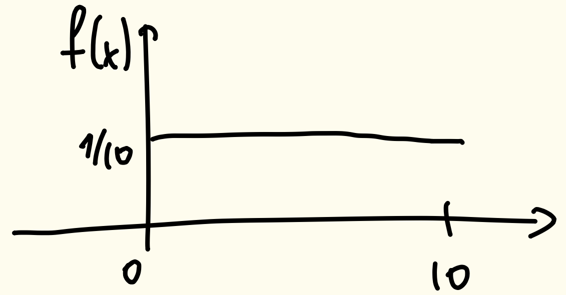
VŠ	<0	<0	<0
ZŠ	>0	>0	<0

$$\operatorname{corr} = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{var}X}\sqrt{\operatorname{var}Y}} \longrightarrow \text{korelace}$$

$$\operatorname{var}X = \mathbb{E}(X-\underbrace{\mathbb{E}X}_{a})^2 = \mathbb{E}\left(X^2-2X\mathbb{E}X-(\mathbb{E}X)^2\right) = \mathbb{E}X^2-\mathbb{E}(2X\underbrace{\mathbb{E}X}_{a})+\mathbb{E}(\underbrace{\mathbb{E}X}_{a})^2$$

$$= \mathbb{E}X^2-\underbrace{2\mathbb{E}X\mathbb{E}X}_{2(\mathbb{E}X)^2}+(\mathbb{E}X)^2 = \underbrace{\mathbb{E}X^2-(\mathbb{E}X)^2}$$

Př. Z doba čekání na bus



$$\operatorname{var}Z = \mathbb{E}Z^2-(\underbrace{\mathbb{E}Z}_{5})^2 \overset{*}{=} \frac{100}{3}-\frac{75}{3}=\frac{25}{3}$$

$$\mathbb{E}Z^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \underbrace{\int_{-\infty}^{0} x^2\, 0\, dx}_{0} + \int_{0}^{10} x^2 \frac{1}{10}\,dx + \underbrace{\int_{10}^{\infty} x^2 \cdot dx}_{0} =$$

$$= \frac{1}{10}\left[\frac{x^3}{3}\right]_0^{10} = \frac{100}{3} \;*$$

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opakování X ... náhodná veličina → charakteristiky:

1) střední hodnota $\mathbb{E}X = \begin{cases} \sum_k k\cdot P(X=k) & \text{diskrétní} \\ \int_{-\infty}^{\infty} x f(x)\,dx & \text{spojité} \end{cases}$

2) rozptyl $\operatorname{var}X = \mathbb{E}(X-\mathbb{E}X) = \mathbb{E}X^2-(\mathbb{E}X)^2$

$\parallel$

$$\begin{cases} \sum_k k^2 P(X=k) \\ \int_{-\infty}^{\infty} x^2 f(x)\, dx \end{cases}$$

Př. Házíme 120× kostkou. Najděte hranici (horní nebo dolní) pro pst, že počet šestek bude větší než 15, ale menší než 25.

X počet šestek ve 120 hodech → $\mathbb{E}X = 20$, var $X =$

$X = \sum_{i=1}^{120} X_i$, kde $X_i = 1$, pokud v i-tém hodu padla šestka
$X_i = 0$ — " — nepadla — " —

$i = 1, \dots, 120$

$\mathbb{E}X_i = 1 \cdot 1/6 + 0 \cdot 5/6 = 1/6$

$\mathbb{E}X_i = 1 \cdot 1/6 + 0 \cdot 5/6 = 1/6$

$\text{var} X_i = \mathbb{E}X_i^2 - (\mathbb{E}X_i)^2 = 1/6 - (1/6)^2 = 5/36$

$\mathbb{E}X_i = 1^2 \cdot 1/6 + 0^2 \cdot 5/6 = 1/6$

$$\mathbb{E}X = \mathbb{E}\sum_{i=1}^{120} X_i = \sum_{i=1}^{120} \mathbb{E}X_i$$

Čebyševova nerovnost pro $\varepsilon = 5$

$$\Rightarrow P(|X-20| \geq 5) \leq \frac{50/3}{25} = 50/75 = 2/3$$

$$1 - P(|X-20| < 5) \leq 2/3$$

$$-P(|X-20| < 5) \leq -1/3 \quad /\cdot(-1)$$

$$P(|X-20|<5) \geq 1/3$$

Př.: kolikrát musíme hodit mincí, aby se podíl orlů lišil od 1/2 o méně než 0,05 s pstí alespoň 0,99.

X = podíl orlů v n hodech $\Rightarrow X = \frac{1}{n}\sum X_i$, kde $X_i = 1$ orel (v i-tém hodu)

$X_i = 0$ panna

$$\Big\} \Rightarrow \mathbb{E}X = \mathbb{E}\, 1/n \cdot \sum_{i=1}^{n} X_i = 1/n \cdot \mathbb{E}\sum_{i=1}^{n} X_i = 1/n \sum_{i=1}^{n} \mathbb{E}X_i = \frac{1}{n}\cdot n \cdot \frac{1}{2} = 1/2$$

$$\operatorname{var}X = \operatorname{var} 1/n \cdot \sum_{i=1}^{n} X_i = 1/n^2 \cdot \operatorname{var}\sum_{i=1}^{n} X_i = \frac{1}{n^2}\cdot \sum_{i=1}^{n} \underbrace{\operatorname{var}X_i}_{1/4} =$$

$$= \frac{1}{n^2}\cdot n \cdot \frac{1}{4} = \frac{1}{4n}$$

$$\mathbb{E}X_i = 1\cdot 1/2 + 0\cdot 1/2 = 1/2, \quad \operatorname{var} X_i = \mathbb{E}X_i^2 - (\mathbb{E}X_i)^2 = 1/2 - 1/4 = 1/4$$

$$\mathbb{E}X_i^2 = 1^2\cdot 1/2 + 0^2 \cdot 1/2 = 1/2$$

$$\text{Č.n.} \Rightarrow P(|X-1/2| \geq 0{,}05) \leq \frac{\frac{1}{4n}}{\left(\frac{1}{20}\right)^2} = \frac{\frac{1}{4n}}{\frac{1}{400}} = \frac{400}{4n} = \frac{100}{n}$$

$(\varepsilon = 0{,}05)$

$$1 - P(|X-1/2| < 0{,}05) \leq \frac{100}{n}$$

$$-P(|X-1/2| < 0{,}05) \leq \frac{100}{n} - \frac{n}{n} = \frac{100-n}{n}$$

$$P(|X-\tfrac{1}{2}| < 0{,}05) \geq \frac{n-100}{n} = 0{,}99 \Leftrightarrow n - 100 = 0{,}99n \Rightarrow$$

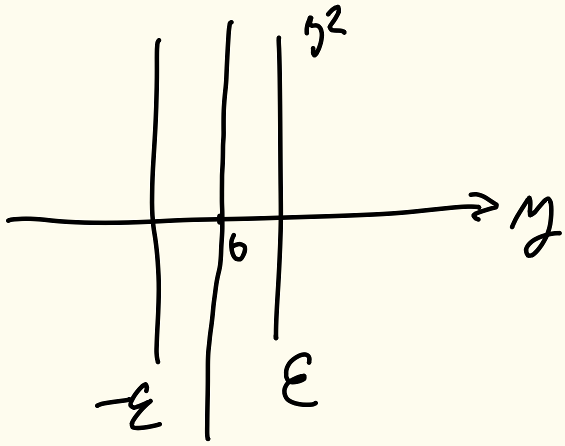
$$\Rightarrow 0{,}01n = 100$$

$$n = 10\,000$$

teoretické průměrování z nabývaných y

$$\int_{-\infty}^{\infty} y^2 \underbrace{dF(y)}$$

$$f(y)\,dy \approx P(Y=y)$$



$$\int_{|y|\geq\varepsilon} y^2\, dF(y) = \int_{|y|\geq\varepsilon} \varepsilon^2\, dF(y)$$

$$= \varepsilon^2 \int_{|y|\geq\varepsilon} dF(y)$$

Příklady náhodných veličin (základní modely)

Př.: X… počet šestek při jednom hodu kostkou $P(X=1)=1/6$

$P(X=0)=5/6$

$$\longrightarrow X \sim Alt(1/6)$$

$$\mathbb{E}X = 1\cdot p + 0\cdot(1-p) = p$$

$$\operatorname{var}X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = p - p^2 = p(1-p)$$

$$\mathbb{E}X^2 = 1^2\cdot p + 0^2(1-p) = p$$

Př. X… počet šestek v pěti hodech kostkou

$$\begin{cases} P(X=0) = \binom{5}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^5 \\ P(X=1) = \binom{5}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^4 \\ P(X=2) = \binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 \end{cases}$$

$$P(X=k) = \binom{5}{k}\cdot\left(\frac{1}{6}\right)^k\cdot\left(\frac{5}{6}\right)^{5-k}$$

pro $k=0,1,\dots,5$

Je $\sum_{k=0}^{5} P(X=k) \stackrel{?}{=} 1$

$$\sum_{u=1}^{5} \binom{5}{u} \underbrace{\left(\frac{1}{6}\right)^{u}}_{a} \underbrace{\left(\frac{5}{6}\right)^{5-u}}_{b} = \left(1/6 + \frac{5}{6}\right)^{5} = 1^5 = 1$$

značíme $X \sim \mathrm{Binom}\left(5; \frac{1}{6}\right)$

Jaká je pst, že šestka padne alespoň 3x?

$$P(X \geq 3) = P(X=3) + P(X=4) + P(X=5)$$

$$\mathbb{E}X = \sum_{k=0}^{n} k \cdot P(X=k) = \sum_{k=0}^{n} k \cdot \underbrace{\frac{n(n-1)!}{k \cdot (k-1)!\,(n-k)!}}_{\frac{n!}{k!\,(n-k)!} = \binom{n}{k}} \cdot \underbrace{p \cdot p^{k-1}}_{p^k} \cdot (1-p)^{n-k} =$$

$$= np \cdot \underbrace{\sum_{k=1}^{n} \underbrace{\frac{(n-1)!}{(k-1)!\,(n-k)!}}_{\binom{m}{i}} p^{k-1} (1-p)^{n-k}}_{} = \begin{pmatrix} i = k-1 \\ m = n-1 \\ m-i = n-k \end{pmatrix} =$$

$n-1+1-k = (n-1)-(k-1)$

Binom věta: $(p + (1-p))^m = 1^m = 1$

$$\operatorname{var} X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \underbrace{\mathbb{E}X^2 - \mathbb{E}X}_{} + \mathbb{E}X - (\mathbb{E}X)^2 = (*)$$

$$\mathbb{E}(X^2 - X) = \mathbb{E}(X(X-1)) = \sum_{k=0}^{n} k(k-1) P(X=k) =$$

$$= \sum_{k=2}^{n} k \cdot (k-1) \frac{n(n-1)(n-2)!}{k(k-1)(k-2)!\,(n-k)!} p^2 p^{k-2} (1-p)^{n-k} =$$

$$= n(n-1)p^2 \cdot \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!\,(n-k)!} \cdot p^{k-2} (1-p)^{n-k} =$$

$$= \left| \begin{matrix} i = k-2 \\ m = n-2 \\ m-i = n-k \end{matrix} \right| = n(n-1)p^2 \sum_{i=0}^{m} \binom{m}{i} p^i (1-p)^{m-i} = n(n-1)p^2$$

$$(*) = n(n-1)p^2 + np - (np)^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p)$$

$$X = \sum_{i=1}^{n} X_i \text{, kde } X_i \sim Alt(p) \Rightarrow \mathbb{E}X = \mathbb{E}\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \mathbb{E}X_i = np$$

$$\operatorname{var} X = \operatorname{var} \sum_{i=1}^{n} X_i \underset{\text{nezávislost}}{=} \sum_{i=1}^{n} \operatorname{var} X_i = np(1-p)$$

$X \sim Binom(n,p)$, přičemž $n \to \infty$, $p \to 0$

$$\mathbb{E}X = np = \lambda \qquad P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} =$$

$$= \frac{n(n-1)\cdot\ldots\cdot(n-k+1)(n-k)!}{k!(n-k)!} p^k \, \underbrace{(1-p)^n}_{} \, (1-p)^{-k}$$

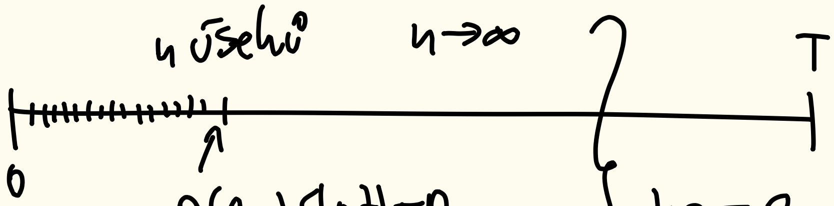
$$\frac{n(n-1)\cdot\ldots\cdot(n-k+1)(n-k)!}{k!(n-k)!} p^k \xrightarrow{n\to\infty} \qquad n^k p^k = \lambda^k$$

$$(1-p)^n = \left(1 - \frac{\lambda}{n}\right)^n \xrightarrow[n\to\infty]{} e^{-\lambda}$$

$$(1-p)^{-k} \xrightarrow{p\to 0} 1$$

$$\xrightarrow[\substack{n\to\infty \\ p\to 0 \\ np=\lambda}]{} \underbrace{\frac{\lambda^k}{k!} e^{-\lambda}}_{} \quad \text{pro } k = 0,1$$

$X \sim Pois(\lambda)$



$P(1 \text{ událost}) = p$

$P(0 \text{ událostí}) = 1-p$ $\Big\}$ $np = \lambda$

$P(2 \text{ a více událostí}) = 0$