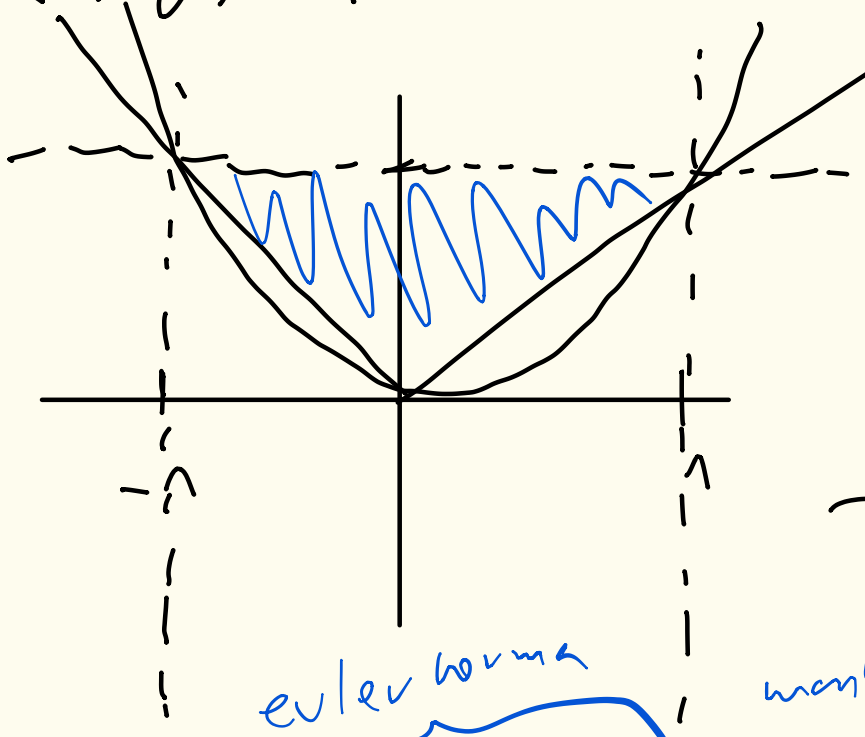
Test CW-1

(1.) (a) $f(x,y) = e^x + |y| - 2$ je konvexní

$e^x \to$ je konvexní
$|y| \to$ je konvexní

součet konvexních fcí je fce konvexní

(b) $\{(x,y) \in \mathbb{R}^2 \mid -1 \le x \le 1,\ x^2 \le y \le 1,\ |x| \le y\}$ je konvex. polyedr



$x^2 \le y$ je redundantní

jedná se o konvexní množinu

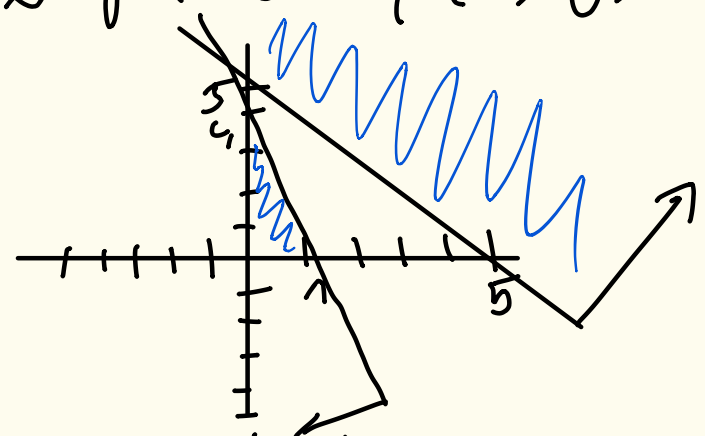
(c) $f(x,y) = \underbrace{\sqrt{x^2+y^2}}_{\text{eukl. norma}} + \underbrace{|x|+|y|}_{\text{manhattan norma}}$ má lok. min, které není glob.

→ součet norem je norma

fce má globální minimum v bodě $(0,0)$, v ostatních bodech je funkční hodnota větší (fce je kostrá)

tedy ne

(d) Konvexní polyedr $\{(x,y) \in \mathbb{R}^2 \mid x+y \ge 5,\ 4x+y \le 4,\ x,y \ge 0\}$ není uzavřená



ano, není uzavřená

(e) každý bod intervalu $[-1,1]$ je regulárním bodem $f(t) = (x-1)^{10}$

fce f je spojitá, v bodě 1 má nulovou derivaci, tedy tento bod není regulární

(2) Matici $A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$ napište ve tvaru SVD

$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$

$$AA^T = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$$

$$\begin{vmatrix} 81-\lambda & -27 \\ -27 & 9-\lambda \end{vmatrix} = (81-\lambda)(9-\lambda) - 27^2 = 0$$

$$(81-\lambda)(9-\lambda) = 27^2$$

$$9 \cdot 81 - 81\lambda - 9\lambda + \lambda^2 = 27^2$$

$$\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 90 \end{bmatrix} \qquad \lambda^2 - 90\lambda = 0$$

$$\lambda(\lambda - 90) = 0 \quad \begin{matrix} 0 \\ 90 \end{matrix}$$

$$\sigma_1 = \sqrt{90} = 3\sqrt{10}$$

$$A^TA - 90I = \begin{bmatrix} -9 & -27 \\ -27 & -81 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

řešíme soustavu

$$\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ řešení $\operatorname{span}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

potom $\vec{v_1} = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

levý singulární vektor

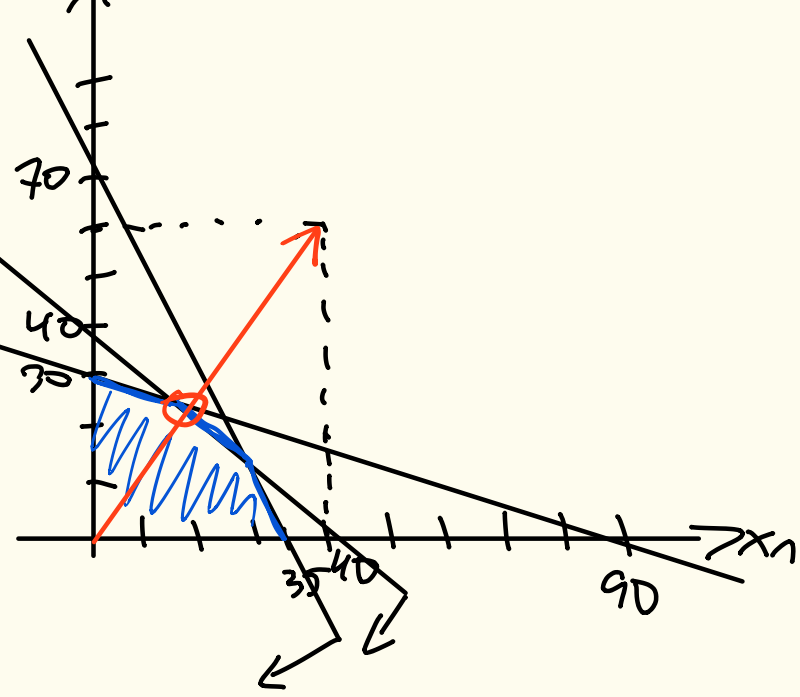
$$u_1 = \frac{1}{\sigma_1} A\vec{v} = \frac{1}{3\sqrt{10}} \cdot \frac{1}{\sqrt{10}} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} =$$

$$= 1/30 \begin{pmatrix} -10 \\ 20 \\ 20 \end{pmatrix}$$

SVD: $3\sqrt{10} \cdot 1/30 \begin{pmatrix} -10 \\ 20 \\ 20 \end{pmatrix} \cdot 1/\sqrt{10} \, (3 \; -1)$

(3) $\max \; 40x_1 + 60x_2$

$$2x_1 + x_2 \leq 70$$
$$x_1 + x_2 \leq 40$$
$$x_1 + 3x_2 \leq 90$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$



$2x_1 + x_2 = 70$

$x_1 + x_2 = 40$

$x_2 = 40 - x_1$

$2x_1 + 40 - x_1 = 70$

$x_1 = 30$

$x_1 + 3x_2 = 90$

$x_1 + x_2 = 40$

$x_1 = 40 - x_2$

$40 - x_2 + 3x_2 = 90$

$40 + 2x_2 = 90$

$x_2 = 25 \quad x_1 = 15$

$(15, 25)$

$(30, 10)$

konvexní polygon s vrcholy:

$(0,30), (15,25), (30,10), (35,0), (0,0)$

maximum v $(15,25)$ hodnota $15 \cdot 40 + 25 \cdot 60 = 2100$

Dual: $\min \; 70y_1 + 40y_2 + 90y_3$

$y_1 \leq 0$

$y_2 \leq 0$

$y_3 \leq 0$

$2y_1 + y_2 + y_3 \leq 40$

$y_1 + y_2 + 3y_3 \geq 60$

Podmínky komplementarity:

$x_1(2y_1+y_2+y_3-40)=0 \quad | \quad x_1=0 \vee 2y_1+y_2+y_3=40$

$x_2(y_1+y_2+3y_3-60)=0 \quad | \quad x_2=0 \vee y_1+y_2+3y_3=60$

$y_1(2x_1+x_2-70)=0 \quad | \quad y_1=0 \vee 2x_1+x_2=70$

$y_2(x_1+x_2-40)=0 \quad | \quad y_2=0 \vee x_1+x_2=40$

$y_3(x_1+3x_2-90)=0 \quad | \quad y_3=0 \vee x_1+3x_2=90$

(4) $f(x,y,z)=3x^2-y^2+3xy+z^2$

(a) $f'(\vec{x})=[6x+3y,\ -2y+3x,\ 2z]$

$\vec{s}=f'\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}=\underline{\underline{2z}}$

$f''(\vec{x})=\begin{bmatrix}6 & 3 & 0\\ 3 & -2 & 0\\ 0 & 0 & 2\end{bmatrix}$

$T_1=f(x_0)+f'(x_0)(x-x_0)$

$=5+\begin{bmatrix}6x_0+3y_0-2y_0+3x_0, 2z_0\end{bmatrix}\begin{bmatrix}x-1\\ y-1\\ z\end{bmatrix}$

$=5+\begin{bmatrix}6+3, 1, 0\end{bmatrix}\begin{bmatrix}x-1\\ y-1\\ z\end{bmatrix}$

$=5+9x-9+y-1=9x+y-5$

(c) $\vec{x}_{k+1} = \vec{x}_k - \alpha_k \nabla f$ pro nějaké α_k

(d) Newton:

$$\vec{x}_{k+1} = \vec{x}_k - \alpha_k (f'')^{-1} \cdot \nabla f \quad \text{pro nějaké } \alpha_k$$

31.5.2023

(1)(a) ano

(b) ne, má jeho nulové

(c) ne, je lineárním

(d) ne

(e) ano

(2) (a) $u = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ $P = \dfrac{uu^T}{u^T u} = \dfrac{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}(1\,1\,2)}{(1\,1\,2)\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} = \dfrac{\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix}}{6} =$

$$1/6 \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1/6 \begin{pmatrix} 4 \\ 4 \\ 8 \end{pmatrix} = \underline{\underline{2/3 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}}$$

(c) $\text{null } A = \text{span}\{(-1, -1, 1)\}$

$$P = \frac{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}(-1, -1, 1)}{(-1,-1,1)\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}} = \frac{\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1/3 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

(e) $\text{null } A^T = \text{span}\{(2,-1,0)\}$

$$A^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$$

$$P = \frac{\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} (2\ -1\ 0)}{(2\ -1\ 0) \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}} = \frac{\begin{bmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}{5} \Rightarrow$$

$$1/5 \begin{bmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} =$$

$$= 1/5 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

(b), (d) ?

(3) Volné extrémy

$$f'(x,y) = (2x+2,\ 2y+2) = \vec{0}$$

$2x+2=0 \quad 2x=-2 \quad x=-1$

$2y+2=0 \quad 2y=-2 \quad y=-1$

podezřelý bod v $(-1,-1)$

$$f''(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$ matice je pos def.

v bodě $(-1,-1)$ tedy je v něm lok. min.

dosazením do podmínek

$1+1 \leq 4$ V

$-3-2 \leq -6$

$-5 \leq -6$ X

nesplňuje, není bod extrémem na množině

Vázané extrémy:

$$h(x,y,\lambda_1) = x^2+2x+y^2+2y+3-\lambda(x^2+y^2-4)$$

$$\frac{\partial h}{\partial x} = 2x+2-2\lambda_1 x = 0 \qquad 2\lambda_1 x = 2x+2$$

$$2\lambda_1 x - 2x = 2$$

$$x(2\lambda_1-2) = 2$$

$$\frac{\partial h}{\partial y} = 2y+2-2\lambda_1 y = 0 \qquad 2\lambda_1 y = 2y+2$$

$$\frac{\partial h}{\partial \lambda_1} = -(x^2+y^2-4) = 0$$

$$\frac{x}{y} = \frac{2x+2}{2y+2} \qquad x = \frac{1}{\lambda_1-1}$$

$$2\lambda_1 y - 2y = 2$$

$$y(2\lambda_1-2) = 2$$

$$y = \frac{1}{\lambda_1-1}$$

$$x^2+y^2-4 = 0$$

$$\left(\frac{1}{\lambda_1-1}\right)^2 + \left(\frac{1}{\lambda_1-1}\right)^2 - 4 = 0$$

$$\frac{2}{(\lambda_1-1)^2} = 4$$

$$1/2 = (\lambda_1-1)^2$$

$$1/2 = \lambda_1^2 - 2\lambda_1 + 1$$

$$\lambda_1^2 - 2\lambda_1 + 1/2 = 0$$

$$\lambda_1 = \frac{+2 \pm \sqrt{4-2}}{2}$$

$$\lambda_1 = \frac{1 \pm \frac{\sqrt{2}}{2}}{2} =$$

$$x = y = \frac{1}{\pm\frac{\sqrt{2}}{2}} = \pm\sqrt{2}$$

$$3\sqrt{2} + 2\sqrt{2} \leq -6 \quad \times$$

$$-3\sqrt{2} - 2\sqrt{2} \leq -6 \quad \checkmark$$

$$f(-\sqrt{2}, -\sqrt{2}) = 7 - 4\sqrt{2}$$

$$\approx 1{,}343$$

loh. min

$$h(x,y,\lambda_2) = x^2+2x+y^2+2y+3-\lambda_2(3x+2y+6)$$

$$\frac{\partial h}{\partial x} = 2x+2-3\lambda_2 = 0 \qquad 2x+2 = 3\lambda_2$$

$$\frac{\partial h}{\partial y} = 2y+2-2\lambda_2 = 0 \qquad 2y+2 = 2\lambda_2$$

$$\frac{\partial h}{\partial \lambda_2} = -(3x+2y+6) = 0$$

$$\frac{2x+2}{2y+2} = 3/2$$

$$2x+2 = 3/2\,(2y+2)$$

$$2x+2 = 3y+3$$

$$3x+2y+6 = 0$$

$$\frac{3(3y+1)}{2}+2y+6=0$$

$$2x = 3y+1$$

$$x = \frac{3y+1}{2}$$

$$3(3y+1)+4y+12=0$$

$$9y+3+4y+12=0$$

$$13y+15=0$$

$$13y = -15$$

$$y = -15/13$$

$$x = \frac{-45/13+1}{2}$$

$$x = \frac{-45/13+13/13}{2}$$

$$x = -16/13$$

$$f(-16/13, -15/13) = 14/13 \longrightarrow \text{lok. min}$$

$$16/13^2 + 15/13^2 \leq 4 \quad \surd$$

$$\approx 2{,}8$$

$$x^2+y^2=4$$

$$3x+2y=-6$$

$$2y=-6-3x$$

$$y=\frac{-6-3x}{2}$$

$$x^2+\left(\frac{-6-3x}{2}\right)^2$$

(4)

Dual

(a) $\min\ 3x_1+x_2+4x_3$

z.p. $x_1+2x_2+ \geq 3$

$x_1 \quad +2x_3 \geq 2$

$2x_1+3x_2+x_3 \geq 4$

$x_1 \geq 0$

$x_2 \geq 0$

$x_3 \geq 0$

$\max\ 3y_1+2y_2+4y_3$

z.p. $y_1 \geq 0$

$y_2 \geq 0$

$y_3 \geq 0$

$y_1+y_2+2y_3 \leq 3$

$2y_1 \quad +3y_3 \leq 1$

$2y_2+y_3 \leq 4$

$(x_1+2x_2-3)\,y_1=0 \rightarrow y_1 \neq 0$

$x_1+2x_2=3$

$x_1+2x_3=2$

$y^*=(1/2, 2, 0)$

$3\cdot 1/2+4 = 3/2+8/2 = 11/2$

$11/2 = x^*\cdot(3,1,4)$

$0+3/2+4x_3=11/2$

$4x_3=8/2$

$x_1=0 \qquad x_3=1$

$3x_1^*+x_2^*+4x_3^*=11/2$

$2x_2=3$

$x^*=(0, 3/2, 1)$

$x_2=3/2$

(d) jedná se o řešení soustavy

$$2y_1 + 3y_3 = 1$$
$$2y_2 + y_3 = 4$$
$$y_3 = 0$$

ano

$$\left(\begin{array}{cc|c} 2 & 0 & 1 \\ 0 & 2 & 4 \end{array}\right)$$

LN sloupce

13.6.2023

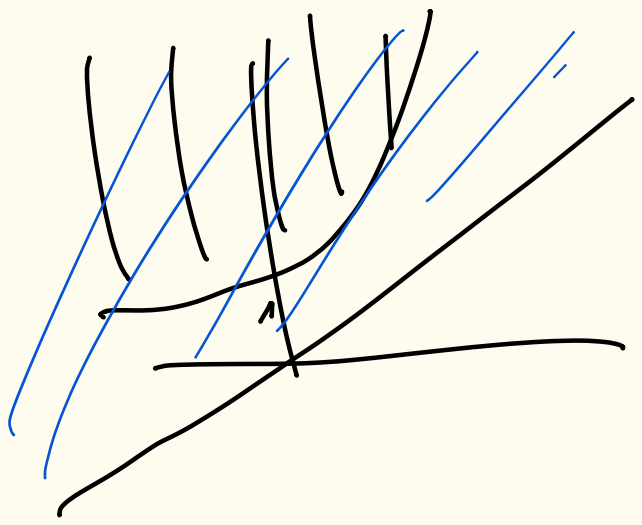
(1) (a) Funkce $f(x,y,z) = |y-z| + |x|$ je konvexní

ano, jedná se o součet konvexních fcí

(b) epigraf $f(x) = e^x$ je konvexní

epigraf $g(x) = x$ je konvexní

průnik konvexních množin je konvexní množina



(průnikem je epigraf e^x)

(c) ano, zmíněná věta o spektrálním rozkladu symetrické AAT

(d) ano, jedná se o matici vlastních čísel, která má vždy hodnost 1

(e) $\min \|Ax-b\|_2^2$

úloha LSQ, ta je konvexní

(2) $X = \text{span}\{(1,2,-1)\} + (0,0,1)$

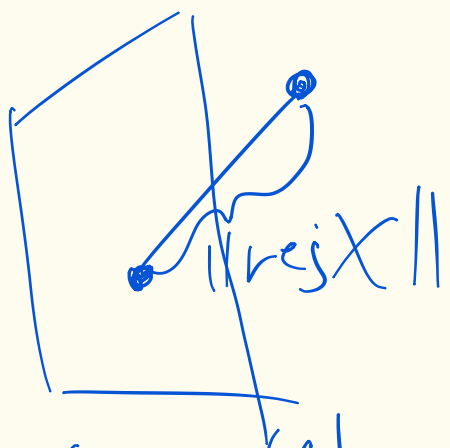
(a)

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftarrow b$$

vzdálenost bodu od množiny

$$\frac{aa^T}{a^Ta} = \frac{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}(1\ 2\ -1)}{(1\ 2\ -1)\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}} = \frac{\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}}{6}$$

$$P_x = 1/6 \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1/6 & 2/6 & -1/6 \\ 2/6 & 4/6 & -2/6 \\ -1/6 & -2/6 & 1/6 \end{pmatrix}$$



$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$P_{x^\perp} = I - P = 1/6 \begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

$$P_{x^\perp} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1/6 \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \quad \left\| \begin{matrix} 1/2 \\ 0 \\ 1/2 \end{matrix} \right\| = \sqrt{2/4} = \frac{\sqrt{2}}{2} = \underline{\underline{1/\sqrt{2}}}$$

(3.) $f(x_0, y_0) = 4$ $\quad (x_0, y_0) = (0, 2)$

Utility:

$$\nabla f(x,y) = \begin{bmatrix} 9/4\,x^2 + 8x + 12y - 28 \\ 12x + 24y - 48 \end{bmatrix}$$

$$\nabla f(x_0, y_0) = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

$$f''(x,y) = \begin{bmatrix} 9/2\,x + 8 & 12 \\ 12 & 24 \end{bmatrix}$$

$$f''(x_0, y_0) = \begin{bmatrix} 8 & 12 \\ 12 & 24 \end{bmatrix} \quad f''(x_0,y_0)^{-1} = 1/48 \begin{bmatrix} 24 & -12 \\ -12 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 1/6 \end{bmatrix}$$

(a) Gradient method

$$\vec{x}_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \alpha \begin{bmatrix} -4 \\ 0 \end{bmatrix} \qquad x = 4\alpha, \quad y = 2$$

$$= \begin{bmatrix} 4\alpha \\ 2 \end{bmatrix}$$

$$\varphi(\alpha) = f(x_1, y_1) = 48\alpha^3 + 64\alpha^2 + 96\alpha + 48 - 112\alpha - 96 + 57$$

$$48\alpha^3 + 64\alpha^2 - 16\alpha + 4$$

$$\varphi(\alpha)' = 144\alpha^2 + 128\alpha - 16$$

$$= 36\alpha^2 + 32\alpha - 4$$

$$= 18\alpha^2 + 16\alpha - 2$$

$$= 9\alpha^2 + 8\alpha - 1$$

$$\alpha_{12} = \frac{-8 \pm \sqrt{64 + 4 \cdot 9}}{18} = \begin{matrix} 1/9 \\ -1 \end{matrix}$$

$$x_{k+1} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} - 1/9 \begin{bmatrix} -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/9 \\ 2 \end{bmatrix}$$

$$f(x_{k+1}, y_{k+1}) = f(4/9, 2) = \frac{748}{243} \approx 3{,}078$$

$$(h)(x_{k+1}, y_{k+1}) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 1/6 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$f(2,1) = 6$$

(c) Ich. minimize nahezu alkohol solvo kinetic
neede to glob. minimum, Surface curvature energy

(4)(a)

Primál

$\min\ 2x_1 - 2x_2$

z.p.

$-x_1 - x_2 \geq -8$

$-4x_1 - x_2 \geq -20$

$-x_2 \geq -6$

$x_1 \geq 0$

$x_2 \geq 0$

Primál

$\min c^T x$

z.p. $Ax \geq b$

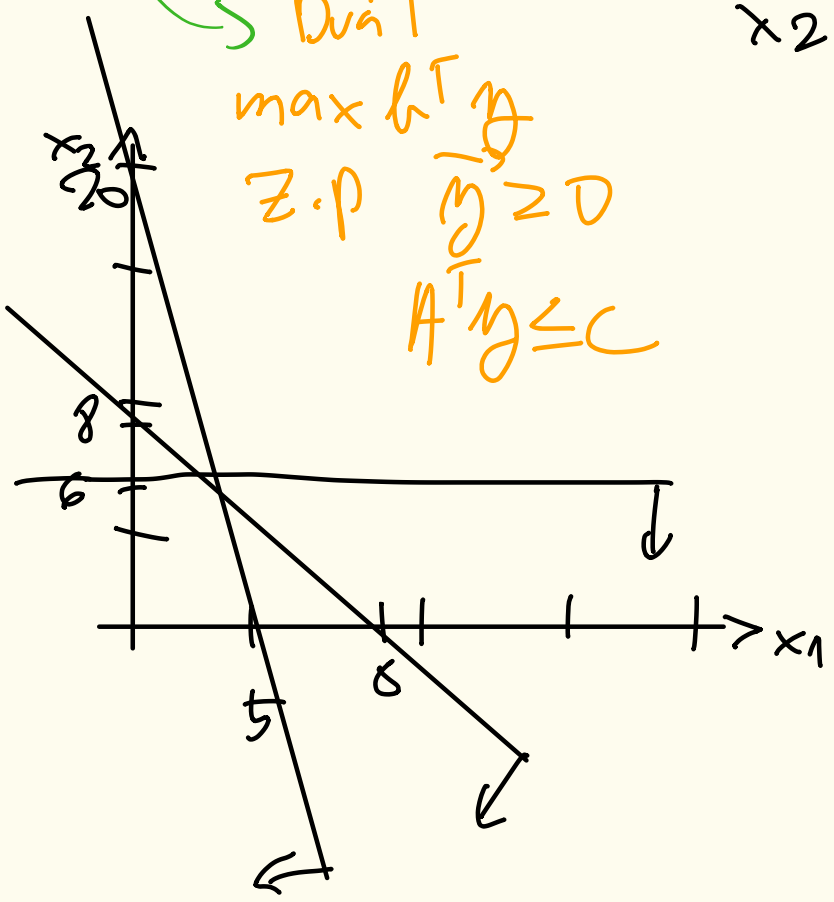
$x \geq 0$

→ Dual

$\max b^T y$

z.p. $y \geq 0$

$A^T y \leq c$



Dual

$\max -8y_1 - 20y_2 - 6y_3$

z.p.

$y_1 \geq 0$ | $x_1 + x_2 = 8 \vee y_1 = 0$

$y_2 \geq 0$ | $4x_1 + x_2 = 20 \vee y_2 = 0$

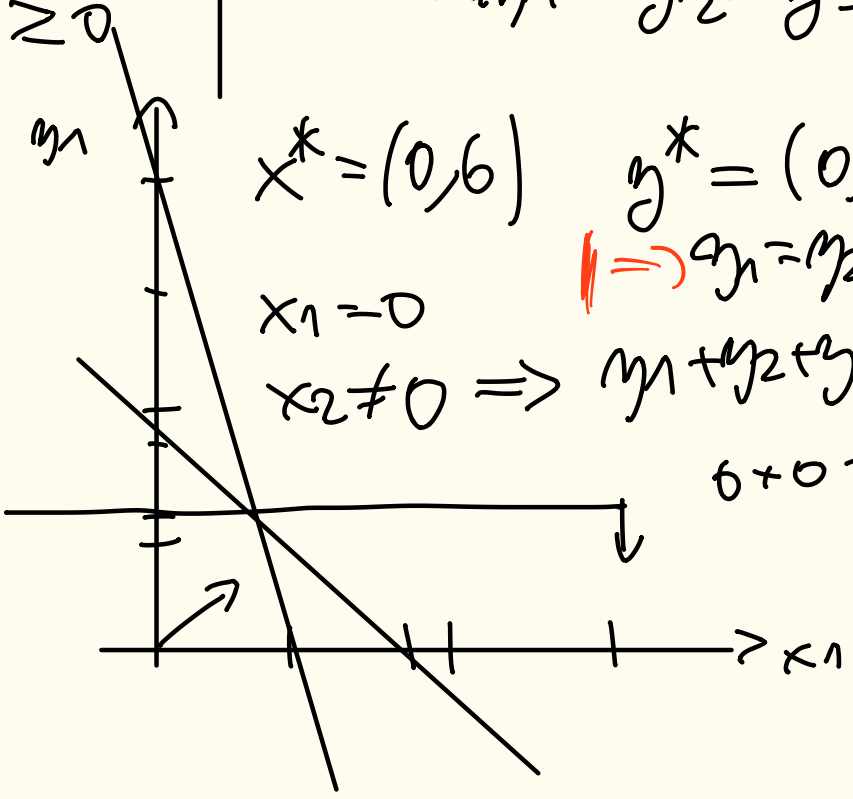
$y_3 \geq 0$ | $x_2 = 6 \vee y_3 = 0$

$x_1 = 0 \vee y_1 + 4y_2 = 2$

$-y_1 - 4y_2 \leq 2$

$x_2 = 0 \vee y_1 + y_2 + y_3 = 2$

$-y_1 - y_2 - y_3 \leq -2$



$x^* = (0,6)$ $y^* = (0,0,2) = -12$

$\Rightarrow y_1 = y_2 = 0$

$x_1 = 0$

$x_2 \neq 0 \Rightarrow y_1 + y_2 + y_3 = 2$

$0 + 0 + y_3 =$

$x_2 = 6$

$x_1 + x_2 = 8$ $(2,6)$ $x_1 + x_2 = 8$

$x_1 = 2$ $4x_1 + x_2 = 20$

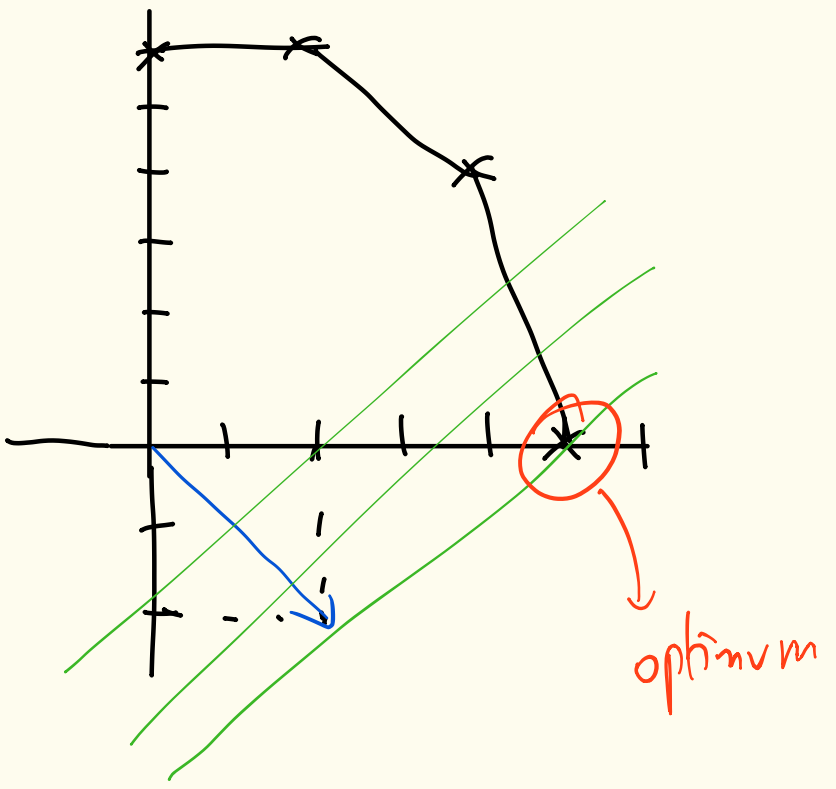
$4x_1 + 8 - x_1 = 20$

$3x_1 = 12$

$x_1 = 4$ $(4,4)$

$(0,6)$ $(5,0)$

$(0,0)$



optimum v $(0,6)$ skutočnou -12

Test 4

(4)

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

matice má řád 4 a hodnost 3,
tedy má 1 vlastní číslo nulové $\Rightarrow$
má jedno smyslami číslo nulové

$$A^TA = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$A^TA = \mathrm{diag}(1,4,9,0) \Rightarrow$ vlastní čísla

$\lambda_4 = 0, \lambda_3 = 1, \lambda_2 = 4, \lambda_1 = 9$

$s_1 = 3, s_2 = 2, s_3 = 1, s_4 = 0$

$A^TA\,v = \lambda v$ $\quad \lambda = 9 \quad \lambda = 4 \quad \lambda = 1 \quad \lambda = 0$

$$\begin{bmatrix} 1 & & & \\ & 4 & & \\ & & 9 & \\ & & & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ a \end{bmatrix} = 9 \begin{bmatrix} x \\ y \\ z \\ a \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

plný sing. v.

$AA^T u = 9u$

$$AA^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix} \quad u = 9u \quad = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

lavé sing. v.

levé: $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

pravé: $\left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array}\right), \left(\begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \end{array}\right), \left(\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array}\right), \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right)$

(d) $\min \{ \|A - B\| \mid B \in \mathbb{R}^{4\times 4}, \operatorname{rank} B \leq 2 \}$

řešením je matice k $n-k$ největších singulárních čísel

$$B^* = \sum M_1 \Delta_1 v_1^T + M_2 \Delta_2 v_2^T = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot 3 \cdot (0\,0\,1\,0) + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot 2 \cdot (0\,1\,0\,0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$x^2+2x+y^2+2y+3$$

$$x^2+y^2 \leq 4$$

$$3x+2y \leq -6$$

---

1) volné extrémy + kontrola, zda splňují podmínky

2) $f+\lambda g_1$ → kontrola, zda splňuje g_2

3) $f+\lambda g_2$ → kontrola, zda splňuje g_1

4) $f+\lambda g_1+\mu g_2$

(1) $f'=0$ $\quad 2x+2=0$

$2y+2=0$

$x=-1$

$y=-1$ $\quad$ mib

$f=1\lambda$

(2) $f+\lambda(x^2+y^2-4)$

$2x+2+\lambda 2x=0$

$2y+2+\lambda 2y=0$

$2(1+\lambda)x=-2$ $\quad x=-\frac{1}{1+\lambda}$

$2(1+\lambda)y=-2$ $\quad y=x$

$x^2+y^2=4$ → $(\sqrt{2},\sqrt{2})\lambda$ → nesplňuje g_2

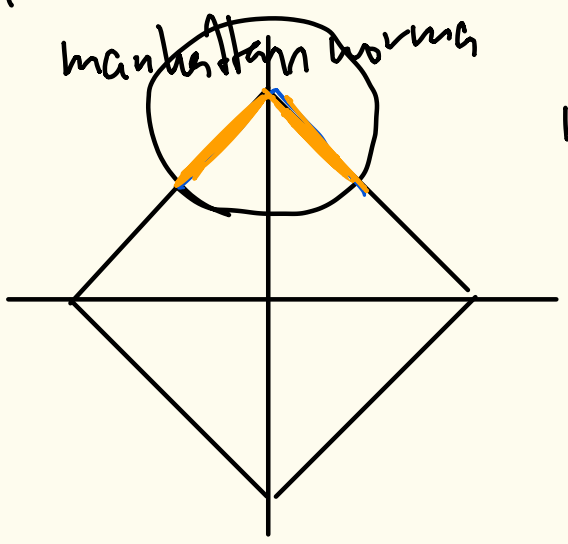
$(-\sqrt{2},-\sqrt{2})$

$$f''(x)=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$$

Test 4

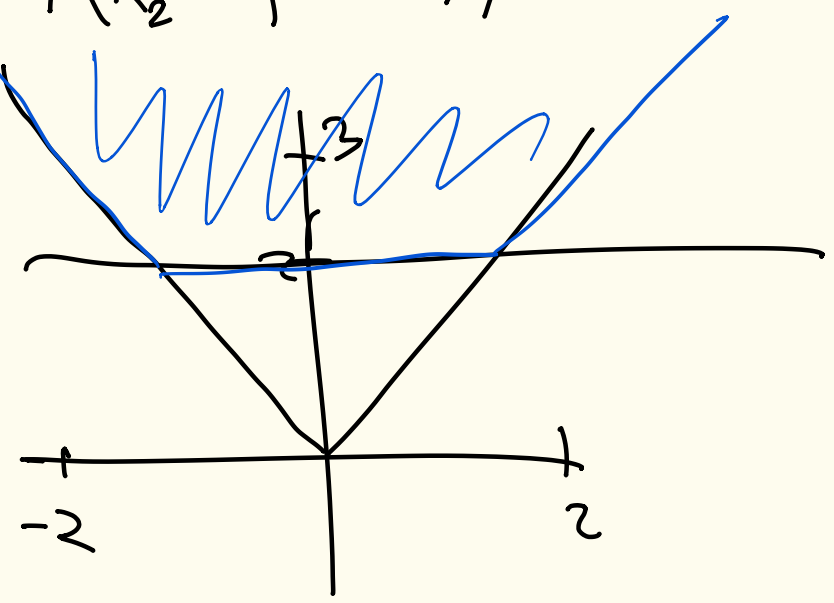
(a) pouze pro positivně definitní A, jedná se o kvadratickou normu F

(b) manhattan norma



horní kružnice, je v podmnožce = F

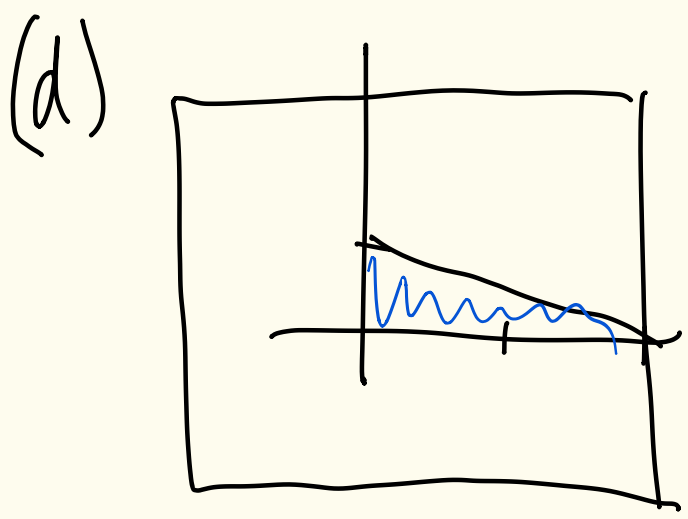
$x_1^2 + (x_2 - 1)^2 \leq 1/4$



(c) $f(x) = \max\{\|x\|_\infty, 2\}$

je konvex T

$-3, -3$



(d) průnik konvexních množin je konvexní T

(e1) omezený konvex $\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$ hledáme LP

(2) Přímka 1 $= \{a_1 + t_1 s_1, t_2 \in \mathbb{R}\}$

Přímka 2 $= \{a_2 + t_2 s_2 \; t_2 \in \mathbb{R}\}$

$\sqrt{\frac{16}{25} + \frac{4}{25}} = \sqrt{\frac{20}{25}} = \sqrt{\frac{4 \cdot 5}{25}}$

$2\sqrt{5}/5 = 2/\sqrt{5}$

hledáme minimum $\|x_1 - x_2\|$ z.p. $x_1 = a_1 + t_1 s_1$ $x_1 = \begin{pmatrix}0\\0\\0\end{pmatrix} + 0\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$x_2 = a_2 + t_2 s_2$ $x_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} - 1/5\begin{pmatrix}0\\1\\-2\end{pmatrix} = \begin{pmatrix}0\\4/5\\2/5\end{pmatrix}$

pravčová soustava 3 rovnicemi a dvěma proměnnými

$t_1, t_2 : a_1 + t_1 s_1 = a_2 + t_2 s_2$

$t_1 s_1 - t_2 s_2 = a_2 - a_1$

$$A \begin{bmatrix} s_1 & -s_2 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} a_2 - a_1 \end{bmatrix}$$

vynásobíme $\begin{bmatrix} s_1^T \\ -s_2^T \end{bmatrix}$ zleva: $\longleftarrow A^T A$

$$A^T A x = A^T b$$

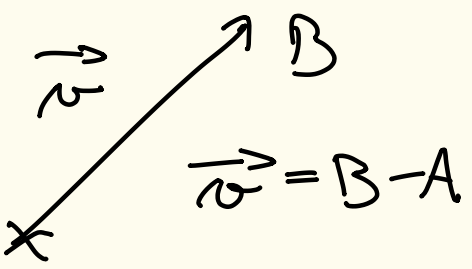
$$\begin{bmatrix} \|s_1\|^2 & -s_1^T s_2 \\ -s_1^T s_2 & \|s_2\|^2 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} s_1^T a_2 - s_1^T a_1 \\ -s_2^T a_2 + s_2^T a_1 \end{bmatrix}$$

$s_1^T a_2$ a $s_1^T a_1$: 0 0; $-s_2^T a_2$: -1; $s_2^T a_1$: 0

1. osa x_1 $\longrightarrow$ $(1,0,0)$ poč. $(0,0,0)$
2. mocha zí body s_1

$(0,1,0 \mid (0,0,2)$

s_2 $(0,1,-2)$



$$\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$t_1 = 0$

$5 t_2 = -1$

$t_2 = -1/5$

(3) $f(x,y) = xy - 3x$

z. p. $x^2 + y^2 \le 9$

$\nabla f(\vec{x}) = \begin{bmatrix} y - 3 \\ x \end{bmatrix}$ $\quad f''(x,y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\det = 0$ indef

vdhé:

$y - 3 = 0 \Rightarrow y = 3$

$x = 0$

$0 + 9 \le 9$ podezřelý bod

je v nom sedlo (indefinitní f'')

vázaná

$$h(x,y,\lambda) = xy - 3x - \lambda(x^2+y^2-9)$$

$\lambda = 0 \quad \frac{\sqrt{3}}{2} \quad \frac{-\sqrt{3}}{2}$

$$\frac{\partial h}{\partial x} = y - 3 - 2\lambda x = 0 \qquad y = 3 + 2\lambda x$$

$$\frac{\partial h}{\partial y} = x - 2\lambda y = 0 \qquad x = 2\lambda y$$

$$\frac{\partial h}{\partial \lambda} = -(x^2+y^2-9) = 0$$

sedlo

$$y = 3 + 2\lambda 2\lambda y$$

$$y = 3 + 4\lambda^2 y$$

$\lambda = 0 \quad x = 0 \quad y = 3$

$\lambda = \frac{\sqrt{3}}{2} \quad x = \frac{3\sqrt{3}}{2} \quad y = -\frac{3}{2}$

$\lambda = \frac{-\sqrt{3}}{2} \quad x = \frac{-3\sqrt{3}}{2} \quad y = \frac{3}{2}$

min

max

$$x = 2\lambda(3 + 2\lambda x)$$

$$x = 6\lambda + 4\lambda^2 x$$

$$y - 4\lambda^2 y = 3$$

$$x - 4\lambda^2 x = 6\lambda$$

$$y(1 - 4\lambda^2) = 3$$

$$x(1-4\lambda^2) = 6\lambda$$

$$y = \frac{3}{1-4\lambda^2} \qquad \frac{3}{1-9\cdot 3}$$

$$x = \frac{6\lambda}{1-4\lambda^2} \qquad \frac{3\sqrt{3}}{-2}$$

$$\left(\frac{6\lambda}{1-4\lambda^2}\right)^2 + \left(\frac{3}{1-4\lambda^2}\right)^2 - 9 = 0 \qquad \lambda \neq \pm 1/2$$

$$\frac{36\lambda^2}{(1-4\lambda^2)^2} + \frac{9}{(1-4\lambda^2)^2} - 9 = 0$$

$$\frac{36\lambda^2 + 9}{(1-4\lambda^2)^2} - 9 = 0$$

$$\frac{36\lambda^2 + 9}{(1-4\lambda^2)^2} - \frac{9(1-4\lambda^2)^2}{(1-4\lambda^2)^2} = 0$$

$$36\lambda^2 + 9 - 9(1-4\lambda^2)^2$$

$$36\lambda^2 + 9 - 9(1 - 8\lambda^2 + 16\lambda^4)$$

$$36\lambda^2 + 72\lambda^2 - 144\lambda^4 = 0$$

$$\lambda^2(36 + 72 - 144\lambda^2) = 0$$

$$36\lambda^2(1 + 2 - 4\lambda^2) = 0$$

$$36\lambda^2(3 - 4\lambda^2) = 0 \begin{cases} \lambda = 0 \\ 4\lambda^2 = 3 \end{cases}$$

$$\lambda^2 = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$$

$A = 1/15 \begin{bmatrix} -13 & 2 & -22 \\ -16 & 14 & -4 \end{bmatrix}$ najděte B hodnosti 1 takovou, že

$\|A-B\|$ je minimální. Pak spočtěte $\|A-B\|$

(a) spektrální rozklad

$$AA^T = 1/15^2 \begin{bmatrix} -13 & 2 & -22 \\ -16 & 14 & -4 \end{bmatrix} \begin{bmatrix} -13 & -16 \\ 2 & 14 \\ -22 & -4 \end{bmatrix} = 1/15^2 \begin{bmatrix} 657 & 324 \\ 324 & 468 \end{bmatrix}$$

$$A = V \Lambda V^T$$

Test 1/2022

(1) (a) $A = \begin{bmatrix} \vec{x} & 1 \end{bmatrix}$ $m = \begin{bmatrix} a \\ b \end{bmatrix}$ $b = \begin{bmatrix} \vec{y} \end{bmatrix}$

úloha je $\min\{\|Am - y\|^2\}$

$$(b) \min \left\| \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\|^2$$

$-z_1 \le b \le z_1$

$\Uparrow$

$|b - 0| \le z_2$

$|a + b - 2|$

$|2a + b - 2|$

$-z_3 \le 2a + b - 2 \le z_3$

$p(x) = x + 1/3$ $z_i \ge 0$, $a \in \mathbb{R}$, $b \in \mathbb{R}$

$$A^T A m = A^T b$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} m = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} b$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} m = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} m = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$5a + 3b = 6$

$3a + 3b = 4$

$2a = 2 \quad a = 1$

$3 + 3b = 4$

$3b = 1$

$b = 1/3$

(c) $\left\| \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1/3 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\|^2 = \begin{bmatrix} 1/3 \\ 4/3 \\ 7/3 \end{bmatrix} - \begin{bmatrix} 0 \\ 6/3 \\ 6/3 \end{bmatrix} = \left\| \begin{bmatrix} 1/3 \\ -2/3 \\ 1/3 \end{bmatrix} \right\|^2$

$\sqrt{\frac{1+4+1}{9}}^2 = 6/9 = \underline{\underline{2/3}}$

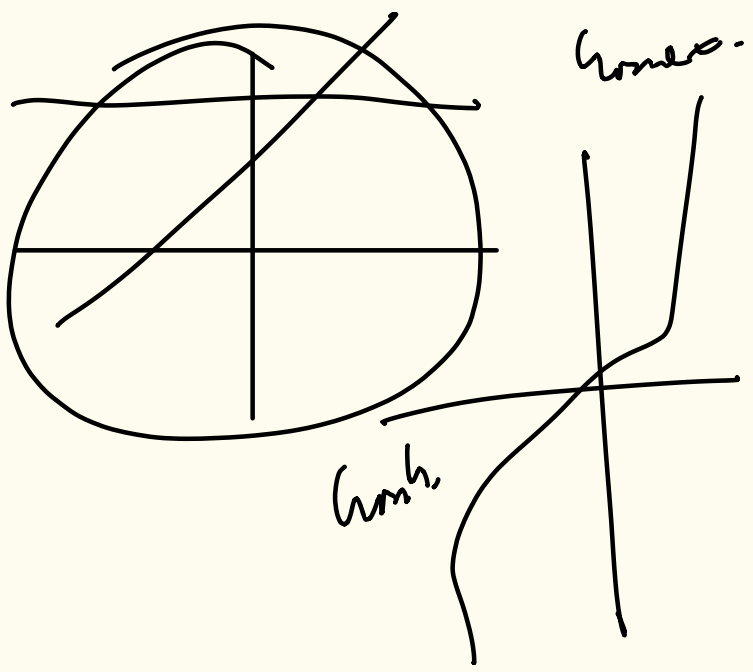
(a) ano

(b) nemusí platit, například pro neg-def. je fce konkávní, ale podmínky platí

(c) je to podmínka homeomorfismu

(d) není $f(x) = 3x^2$ $f'(1) = 6 \neq 0$

(e) není homeom.



Hess.

$\nabla f \begin{pmatrix} e^{x_1} \\ -2x_2 \end{pmatrix}$

$Hf = \begin{bmatrix} e^{x_1} & 0 \\ 0 & -2 \end{bmatrix}$ $\Delta_1 = +$ $\Delta_2 = -$ indef.

(3) $f(x) = x^2 - 1/2 x^4$ na $\mathbb{R}$

(a) $f'(x) = 2x - 2x^3$ $\quad 2x(1-x^2) = 0 \begin{cases} x = 0 \\ x = \pm 1 \end{cases}$

$f''(x) = 2 - 6x^2$ $\quad f''(0) = 2$ loh min

$f''(1) = -4$ loh max

$f''(-1) = -4$ loh max

(b) grad $x_{k+1} = x_k - \alpha \nabla f(x_k) = x_k - 2\alpha_k x_k + 2\alpha_k x_k^3$

Newton $= x_k - \frac{f'(x_k)}{f''(x_k)} = \frac{x_k - 2x_k - 2x_k^3}{2 - 6x_k^2} = \frac{x_k - x_k - x_k^3}{1 - 3x_k^2} = \frac{-2x_k^3}{1-3x_k^2}$

(c) $x_{n+1} = x_n^3$ — konvergira pro $(-1, 1)$ ∂x

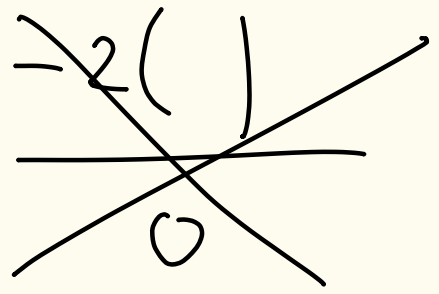
pro 1, −1 konvergence +h

(ad) $x_{n+1} = x_n^2$

Konv. $1/2 \to 1/4 \to 1/16 \to 1/256 \to$ lim $\to 1/\infty = 0$

div. $2 \to 4 \to 16 \to 256 \to \infty$

(d) $\pm\frac{\sqrt{3}}{3}$ →



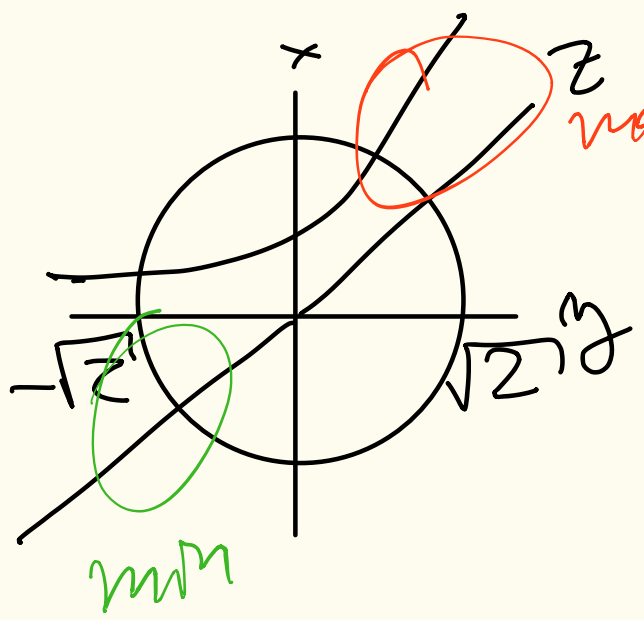
$$\frac{-2x_1^3}{1-3x_1^2} = 0$$

$$-2x_n^3 = 0$$

$$\Downarrow$$

pouze pro $x_n = 0$

(4)



$\to$ $f(x,y) = xe^y \to$

je nespojitá

maximum i minimum bude na obvodových hranicích

(max)

min ve 3

$$h(x,y,\lambda) = xe^y - \lambda(x^2+y^2-2)$$

$$\frac{\partial h}{\partial x} = e^y - 2\lambda x = 0 \qquad e^y = 2\lambda x$$

$$\frac{\partial h}{\partial y} = xe^y - 2\lambda y = 0 \qquad xe^y = 2\lambda y$$

$$\frac{\partial h}{\partial z} = -(x^2+y^2-2) = 0 \qquad x2\lambda x = 2\lambda y$$

$$x^2 = y$$

$$y + y^2 - 2 = 0$$

$$y^2 + y - 2 = 0$$

$$y_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = \begin{cases} -2 & \times \\ 1 & (1,1) \end{cases}$$

$$x^2 = y \qquad x = \pm 1$$

$(-1,1) \quad (1,1)$

min max

(5)

Primární	Duální
max $c^T x$	min $d^T y$
z.p. $x \le d$	z.p. $y \ge 0$
$x \ge 0$	$y \ge c$

$x_i = 0$ pro $c_i < 0$

$x_i = d$ pro $c_i \ge 0$

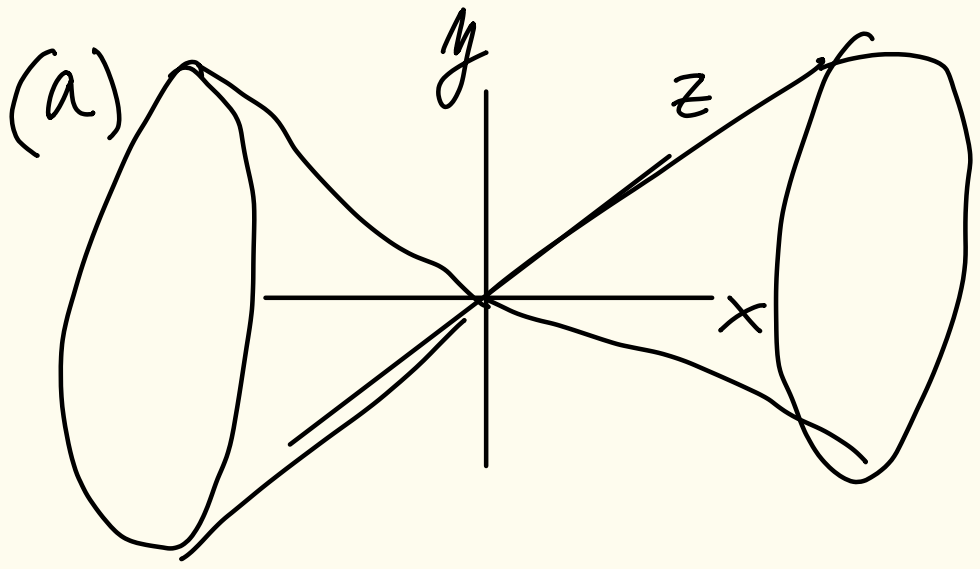
$x_i (y_i - c_i) = 0$

$y_i (x_i - d_i) = 0$

nerovnice $d \ge 0$

a $\exists\, d_i < 0$

$\Rightarrow$ nepřípustná $\longrightarrow$ neomezená



(b) $\min \|x - (1,3,4)\|$

$x \in M$

M není konvex $\Rightarrow$ není úloha konvex. optimalizace

(c) $g(x) = x^2 = y^2 + z^2$

$$f(x) = (x-1)^2+(y-3)^2+(z-4)^2$$

$$L(x,y,z,\lambda) = (x-1)^2+(y-3)^2+(z-4)^2-\lambda(y^2+z^2-x^2)$$

$$\frac{\partial h}{\partial x} = 2x-2+2\lambda x = 2x+2\lambda x=2 \quad 2x(1+\lambda)=2 \quad x=\frac{1}{1+\lambda}$$

$$\frac{\partial h}{\partial y} = 2y-6-2\lambda y = 2y-2\lambda y = 6\ 3 \quad 2y(1-\lambda)=6 \quad y=\frac{3}{1-\lambda}$$

$$\frac{\partial h}{\partial z} = 2z-8-2\lambda z = 2z-2\lambda z = 8 \quad 2z(1-\lambda)=8\ 4 \quad z=\frac{4}{1-\lambda}$$

$$\frac{\partial h}{\partial \lambda} = -(y^2+z^2-x^2) =$$

$$\left(\frac{3}{1-\lambda}\right)^2+\left(\frac{4}{1-\lambda}\right)^2-\left(\frac{1}{1+\lambda}\right)^2=0$$

$$\frac{9}{(1-\lambda)^2}+\frac{16}{(1-\lambda)^2}-\frac{1}{(1+\lambda)^2}=0$$

$$\frac{25}{(1-\lambda)^2}=\frac{1}{(1+\lambda)^2}$$

$$25(1+\lambda)^2=(1-\lambda)^2$$

$$25(1+2\lambda+\lambda^2)=(1-2\lambda+\lambda^2)$$

$$25+50\lambda+25\lambda^2=1-2\lambda+\lambda^2$$

$$24+52\lambda+24\lambda^2=0$$

$$6\lambda^2+13\lambda+6=0$$

$$\lambda_{12}=\frac{-13\pm\sqrt{13^2-4\cdot 36}}{12} \begin{cases} -2/3 \\ -3/2 \end{cases}$$

	$-2/3$	$-3/2$
x	3	-2
y	$9/5$	$6/5$
z	$12/5$	$8/5$

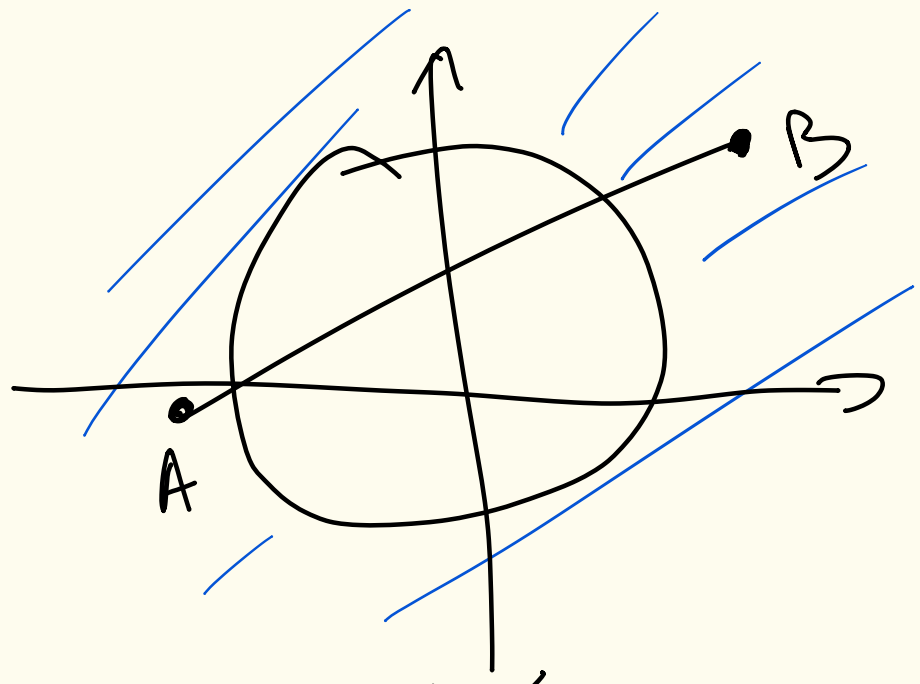
vzdálenosti

$(3, 9/5, 12/5)$ $2\sqrt{2}$

$(-2, 6/5, 8/5)$ $3\sqrt{2}$

$(x-1)^2+(y-3)^2+(z-4)^2$

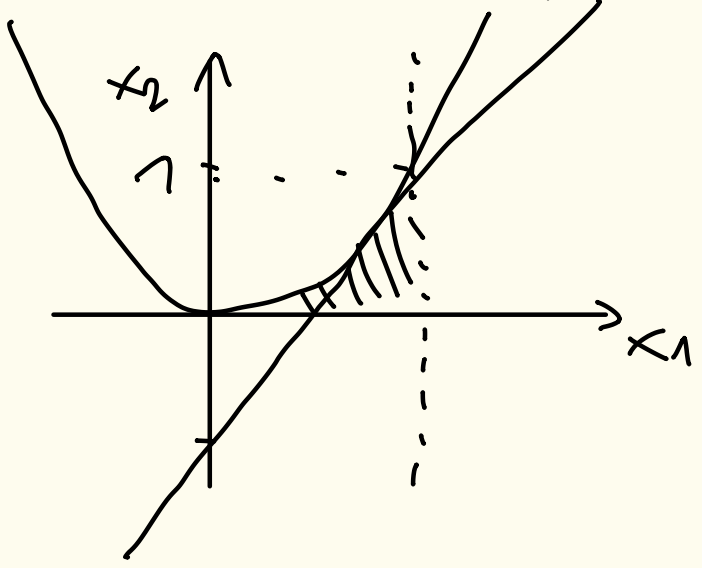
(a)



není konvex

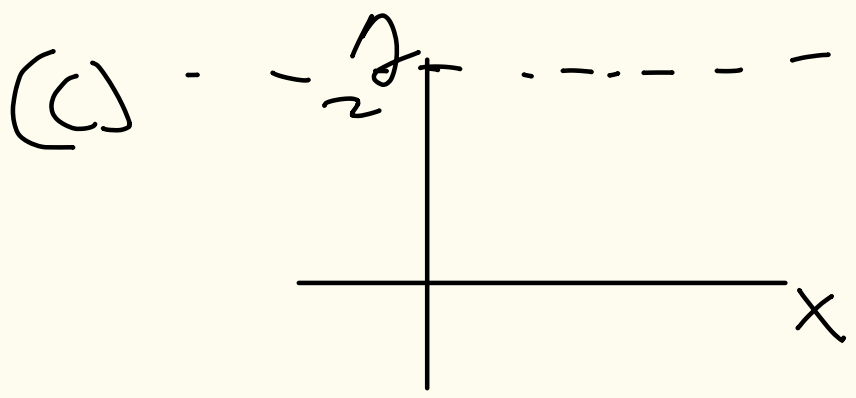
AB není vrcholů K

(b)



průnik konvex. množin, ten je konvex.

(c)



je sublevelová množina součtu konvex
⇒ je konvex

(d) $Ax - b = 0 \rightarrow$ aff. podp.
$x \geq 0$ konvex.
je konvexní

(e) $\nabla f(x,y) = \begin{bmatrix} 2x+2y \\ 2y+2x \end{bmatrix}$

$H_f(x,y) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ $\Delta_1 = +$, $\Delta_2 = 0$ positivně semidef $\Rightarrow$ konvexní

(2) $f(x_1,x_2) = (x_1,x_2) A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + b^T x + c$

$A = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 3 \end{pmatrix}$ $b = \begin{pmatrix} 0 \\ 11 \end{pmatrix}$ $c = 11$

$$\nabla f(x_1,x_2) = \begin{bmatrix} 2x_1 + x_2 \\ 6x_2 + x_1 + 11 \end{bmatrix}$$

$$Hf(x_1,x_2) = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix} \quad \begin{matrix} \sigma_1 = + \\ \sigma_2 = + \end{matrix}$$

$2x_1 + x_2 = 0 \qquad x_2 = -2x_1$

$6x_2 + x_1 = -11 \qquad x_2 = -2$

$-12x_1 + x_1 = -11$

$-11x_1 = -11$

$x_1 = 1$

extrem v (1,2)
loh. min.

Taylor 1

$$T_1(0,1) = f(0,1) + f'(0,1)\begin{pmatrix} x_1 - 0 \\ x_2 - 1 \end{pmatrix}$$

$$= 3 + 11 + 11 + \begin{bmatrix} 1 \\ 17 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 - 1 \end{pmatrix} = x_1 + 17x_2 + 8$$

$\alpha = 1 \quad \beta = 17 \quad \gamma = 8$

Test 3)

$x = (1,1,1,1)$

$y = (1,0,-1,0)$

$X = \operatorname{span}\{x,y\}$

$X^{\perp} = \operatorname{span}\left(\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}\right)$ orthonorm. Säge: $\operatorname{span}\left(\frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}\right)$

(d) UU^T

$$U = \begin{bmatrix} 1/2 & 0 \\ -1/2 & -\frac{\sqrt{2}}{2} \\ 1/2 & 0 \\ -1/2 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$UU^T = \begin{bmatrix} 1/2 & 0 \\ -1/2 & -\frac{\sqrt{2}}{2} \\ 1/2 & 0 \\ -1/2 & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 & 1/2 & -1/2 \\ 0 & -\sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1/4 & -1/4 & 1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 & -2/4 \\ 1/4 & -1/4 & 1/4 & -1/4 \\ -1/4 & -1/4 & -1/4 & 3/4 \end{bmatrix} = 1/4\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 3 & -1 & -2 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

(c) $I - UU^T$

(2)

$A \in \mathbb{R}^{m \times n}$ $m \geq n$, sloupce

$A \in \mathbb{R}^{n \times m}$ $a \in \mathbb{R}^n$

$B \in \mathbb{R}^{n \times p}$

$C \in \mathbb{R}^{n \times q}$

$D \in \mathbb{R}^{q \times p}$ $p_n = q_r$

$x \in \mathbb{R}^m$

$y \in \mathbb{R}^p$

$z \in \mathbb{R}^q$

$u \in \mathbb{R}^q$

$$\begin{bmatrix} A & B & -C \\ 0 & D & I \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ u \end{bmatrix}$$

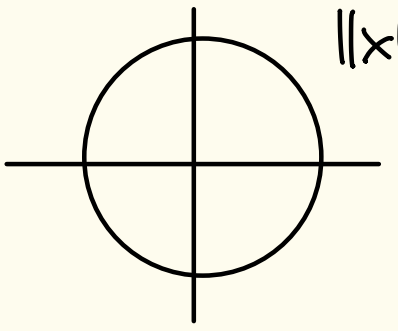
$Ax + By = a + Cz$

$Dy + z = u$

ano, pokud by matice P byla regulární $(m + p = n)$

3. Rozhodněte, zda je zadaný optimalizační problém konvexní a odpověď zdůvodněte.
   (a) (2 b) Minimalizuj $\mathbf{x}^T\mathbf{A}\mathbf{x}$ za podmínek $\|\mathbf{x}\|^2 = 1$, $\mathbf{x} \in \mathbb{R}^n$, kde $\mathbf{A} \in \mathbb{R}^{n \times n}$ je symetrická.
   (b) (2 b) Minimalizuj $x + y$ za podmínek $y \leq x^2$, $(x, y) \in \mathbb{R}^2$.
   (c) (2 b) Minimalizuj $3x - y + z$ za podmínek $x + y - z \leq 2$, $y - z \geq -1$, $x, y, z \geq 0$.
   (d) (2 b) Minimalizuj $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|$ za podmínky $\mathbf{x} \in \mathbb{R}^n$, kde $\|.\|$ je norma, $\mathbf{A} \in \mathbb{R}^{m \times n}$ a $\mathbf{b} \in \mathbb{R}^m$.
   (e) (2 b) Maximalizuj $x^2 + y^2$ za podmínek $x + y \leq 1$, $x, y \geq 0$.

(a)



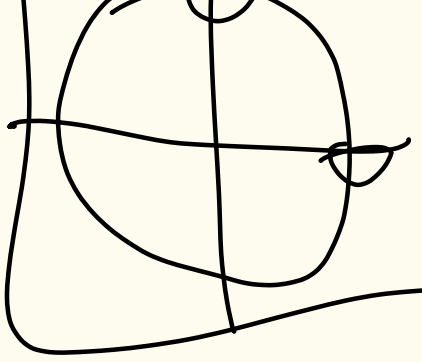
$\|x\|^2 = 1$ není konvex.

(e) $\max x^2 + y^2$ není LP

$2p.\ x + y \leq 1$

$x, y \geq 0$

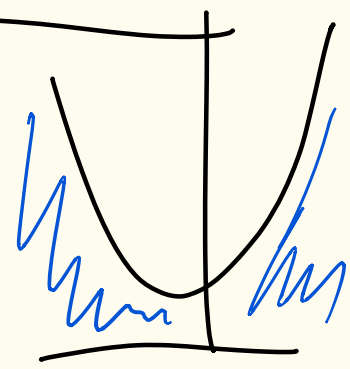
není konvex.



(b) $x + y$ je lineární

$y \leq x^2$ je epigraf x^2, což není konvexní množina

nejedná se o problém konvexní optimalizace



(c) $\min 3x - y + z$

$2p.\ x + y - z \leq 2$

$y - z \geq -1$

$x, y, z \geq 0$

jedná se o LP

jedná se o optimalizaci na konvexním polyedru

(d) $\min \|Ax-b\|$ z.p. $x \in \mathbb{R}^n$

$\|.\|$ je konvex

$Ax-b$ je duhý konvex $\Rightarrow$ konvex

Primál

$\max \; 2x_1 + 5x_2$

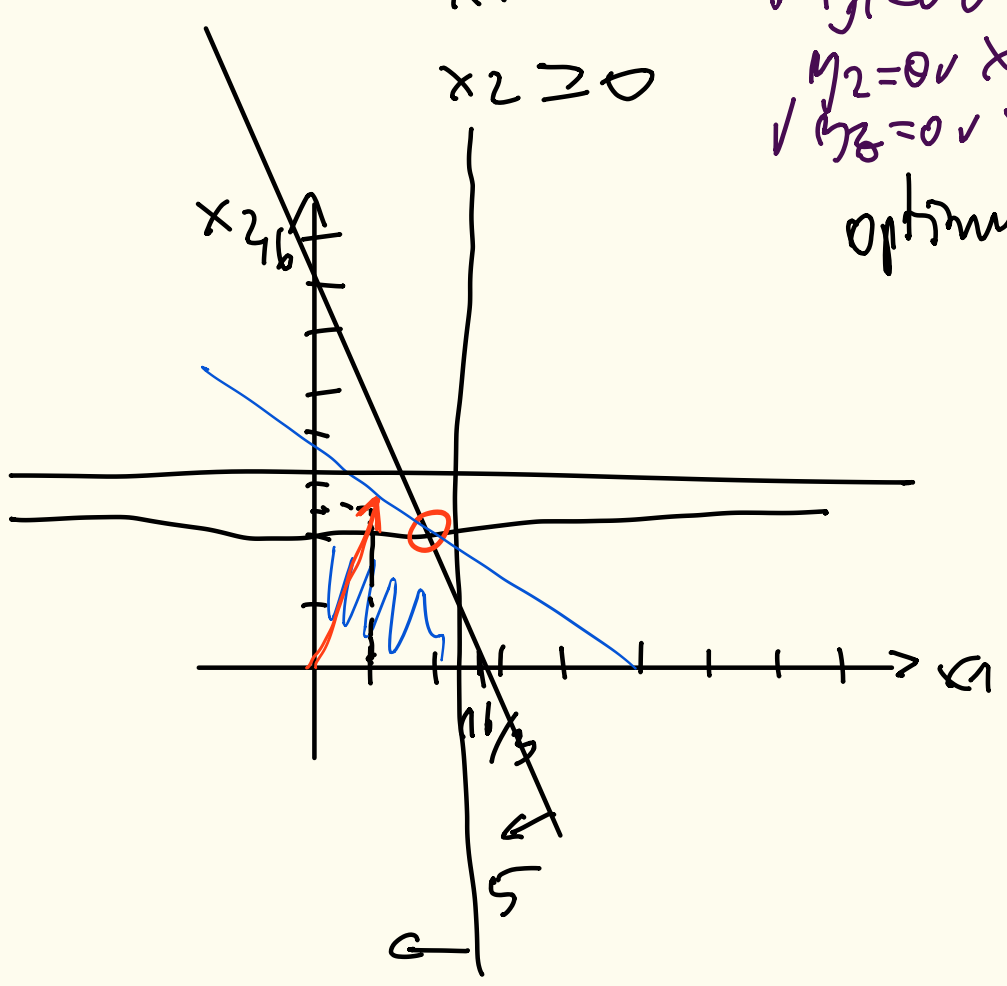
z.p. $3x_1 + x_2 \leq 16$

$x_1 \leq 5$

$x_2 \leq 4$

$x_1 \geq 0$

$x_2 \geq 0$



Duál

$\min \; 16y_1 + 5y_2 + 4y_3$

z.p.

$y_1 \geq 0$

$y_2 \geq 0$

$y_3 \geq 0$

$3y_1 + y_2 \geq 2$

$y_1 + y_3 \geq 5$

$x_1 = 0 \vee 3y_1 + y_2 = 2 \; V$

$x_2 = 0 \vee y_1 + y_3 = 5 \; V$

$\sqrt{} \; y_1 = 0 \vee 3x_1 + x_2 = 16 \; V$

$y_2 = 0 \vee x_1 = 5 \; \times$

$\sqrt{} \; y_3 = 0 \vee x_2 = 4 \; V$

optimum stejné $\Leftarrow$ věta o silné dualitě

optimum v bodě:

$3x_1 + x_2 = 16$

$x_2 = 4$

$3x_1 + 4 = 16$

$3x_1 = 12$

$x_1 = 4$

$(4,4) = x^*$

$8 + 20 = 28$

---

(5)

(a)

$$f(x) = \sum_{i=1}^{n} \left(a_i^T x - y_i \right)^2$$

$\min \|Ax-y\|^2 \Rightarrow Rank(A) = n$

(b) první derivace

$$\sum_n 2\left(a_i^T x - y_i\right) a_i^T = 2(Ax-y)^T A$$

druhá derivace:

$$\sum 2 a_i a_i^T = 2A^T A$$

$x_{k+1} = x_k - 2\alpha (Ax-y)^T A$

Newton

$x_{k+1} = x_k - (2A^TA)^{-1} 2A^T (Ax_k - y)$

$$= \boxed{(A^TA)^{-1}A^Ty} \Rightarrow LSB$$

$x = (1,0,-2) \in \mathbb{R}^3$

$y = (2,0,1)$

$$A = \left[\frac{x}{\|x\|}, \frac{y}{\|y\|}\right] \in \mathbb{R}^{3\times 2}$$

(a) $AA^T = UU^T \rightarrow$ ortog. projektor T

(b) $\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 0 & 0 \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$AA^T = \text{diag}(1,0,1)$

ano má vlastní číslo 1

(c) není, není čtvercová

(d) $f(z) = \|Az\|^2$ ano, jde o složení dvou lineárních funkcí

(e) není, není LN sloupce

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 6 \\ 6 & 4 \end{bmatrix}$$

$$\begin{vmatrix} 13-\lambda & 6 \\ 6 & 4-\lambda \end{vmatrix} \rightarrow (13-\lambda)(4-\lambda) - 36 = 0$$

$\sigma_1 = 4$

$\sigma_2 = 1$

$$\begin{bmatrix} -3 & 6 \\ 6 & -12 \end{bmatrix} u_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} / \sqrt{5}$$

$$52 - 4\lambda + \lambda^2 - 8\lambda - 36 = 0$$

$$\lambda^2 - 12\lambda + 16 = 0$$

$$\lambda_{12} = \frac{12 \pm \sqrt{12^2 - 4 \cdot 16}}{2} \begin{cases} 16 \\ 1 \end{cases}$$

$$\begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix} u_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} / \sqrt{5}$$

$$v_1 = \frac{1}{\sigma_1} u_1^T A = \frac{1}{\sqrt{5}} (2\ 1) \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} / \sqrt{5} \cdot 1/4$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} / \sqrt{5}$$

$$v_2 = \frac{1}{\sigma_2} u_2^T A = 1/\sqrt{5} (1\ -2) \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Primár

(3) $\min -x_1 + 2x_2$

z.p. $x_1 + x_2 \leq 4$

$2x_1 + x_2 \leq 5$

$-x_1 + 4x_2 \geq 2$

$x_1 \geq 0$

$x_2 \geq 0$

Duál

$\max 4y_1 + 5y_2 + 2y_3$

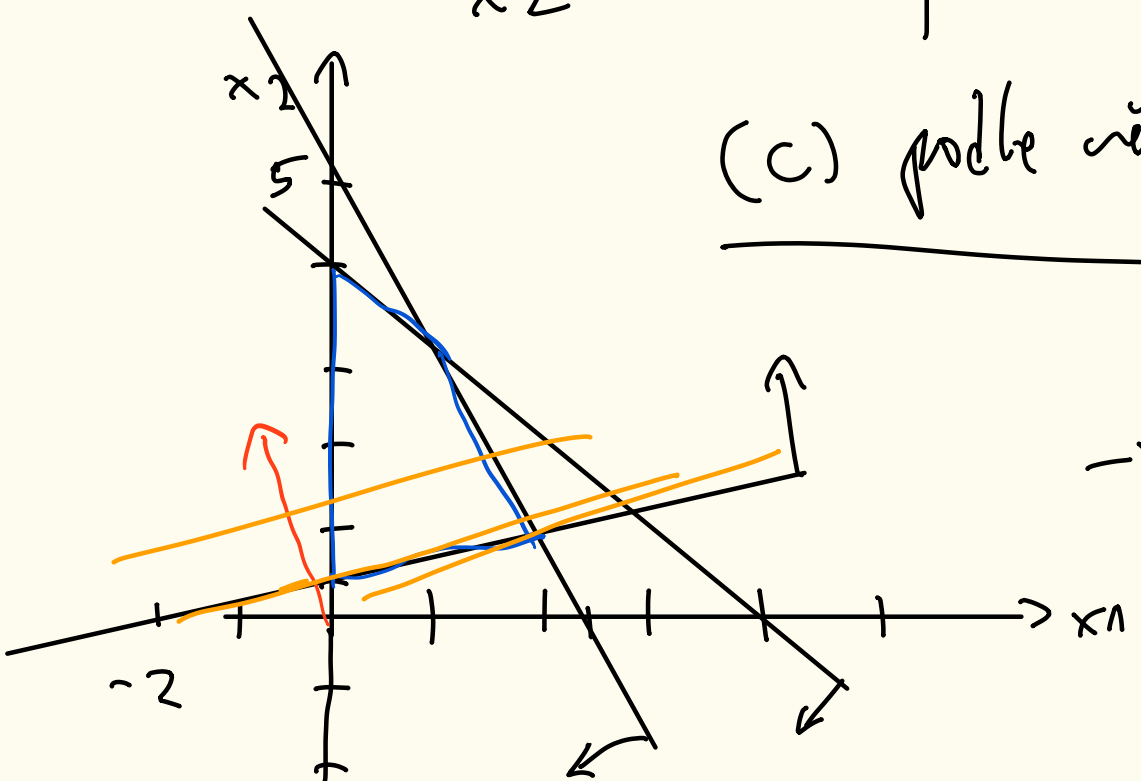
$y_1 \leq 0$

$y_2 \leq 0$

$y_3 \geq 0$

$y_1 + 2y_2 - y_3 \leq -1$

$y_1 + y_2 + 4y_3 \leq 2$



(c) podle věty o silné dualitě je optimum duální úlohy stejné jako primární

$x_1 = 4x_2 - 2$

$-x_1 + 4x_2 = 2$

$2x_1 + x_2 = 5$

$2(4x_2 - 2) + x_2 = 5$

$8x_2 - 4 + x_2 = 5$

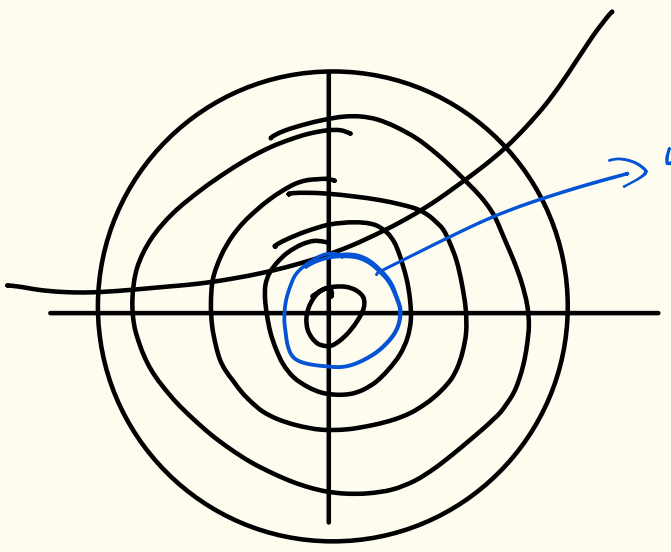
$9x_2 = 9 \quad x_1 = 4 - 2$

$x_2 = 1 \quad x_1 = 2$

optimum: $\underline{\underline{(2,1)}}$

optimální hodnota je 0

(4) $f(x,y) = x^2 + y^2$ na mn. $\{(x,y) \in \mathbb{R}^2,\ y = e^x\}$



(g je dedná na $f(0,y)$

vázané extrémy: $\nabla f \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ $\quad 2x = 0$ $\quad 2y = 0$ $\quad 0 \neq e^0$ ↯

vázané extrémy

$h(x,y,\lambda) = x^2 + y^2 - \lambda(e^x - y)$

$\frac{\partial h}{\partial x} = 2x - \lambda e^x = 0$

$\frac{\partial h}{\partial y} = 2y + \lambda = 0$

$\frac{\partial h}{\partial \lambda} = -(e^x - y) = 0$

$2x = \lambda e^x$

$e^x = \frac{2x}{\lambda}$

$2y = -\lambda$

$y = -\frac{\lambda}{2}$

$\frac{2x}{\lambda} + \frac{\lambda}{2} = 0$

$e^x = y$

$2x + 2y e^x = 0$

$x + y e^x = 0$

$x + e^x e^x = 0$

$$g(x) = x + e^{2x} = 0$$

$$x + (e^x)^2 \Rightarrow x + e^{2x} = 0$$

Newtons μέθοδος:

Gradient descent

$$g'(x) = 1 + 2e^{2x}$$

$$x_{n+1} = x_n - \lambda\left(1 + 2e^{2x_n}\right)$$

$$g''(x) = 4e^{2x}$$

για $\lambda > 0$

$$x_{n+1} = x_n - \frac{1}{4e^{2x_n}} \cdot \left(1 + 2e^{2x_n}\right)$$

$\neq 0$ για $x \in \mathbb{R}$

(5)

(a) $$\nabla f = \begin{bmatrix} 2x + 2y \\ 2y + 2x \\ 2z \end{bmatrix}$$

$$\text{dir. } \vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underline{2x + 2y}$$

(b) $$T_1(1,1,1) = 5 + \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}^T \begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix} =$$

$$= 5 + 4x - 4 + 4y - 4 + 2z - 2 =$$

$$= \underline{4x + 4y + 2z - 5}$$

(c)

$$f(x,y,z) = x^2 + 2xy + y^2 + z^2$$

$$= (x+y)^2 + z^2 \Rightarrow \text{convex}$$

convex convex

(d) $(x+y)^2 + z^2 = 1$ cylindrique