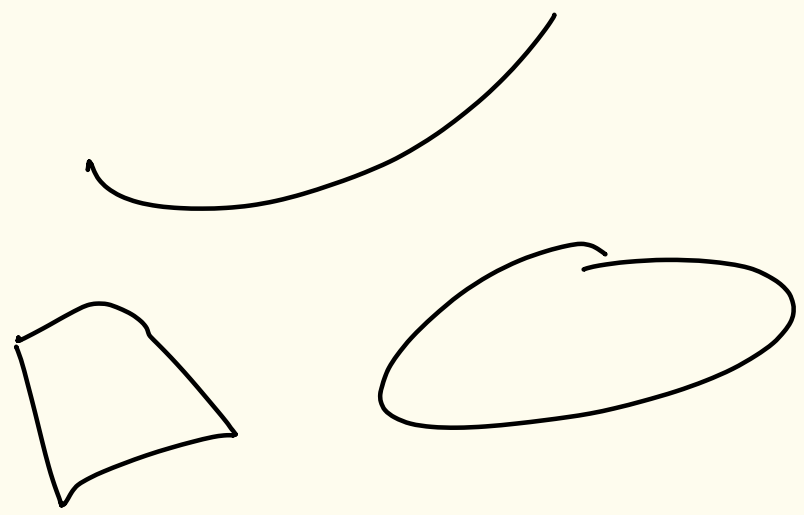
Konvexita

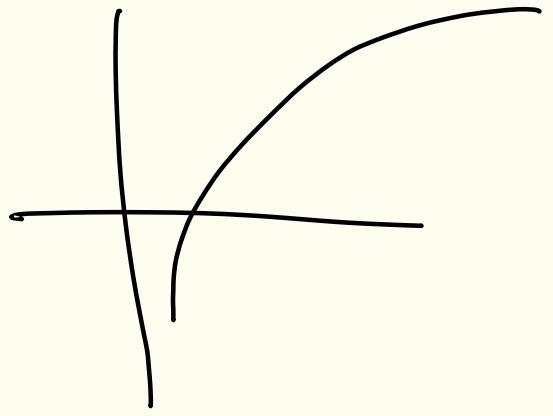
- množiny
- funkce



$$f\left((1-\alpha)x + \alpha y\right) \leq (1-\alpha) f(x) + \alpha f(y)$$

$f(x) = \ln x$, $x > 0$ konkávní

(2. der $-\frac{1}{x^2}$)



$x_1, \dots, x_n > 0$, $\alpha_1 = \dots = \alpha_n = 1/n$

$$f\left(\frac{x_1 + \dots + x_n}{n}\right) \geq \frac{f(x_1) + \dots + f(x_n)}{n}$$

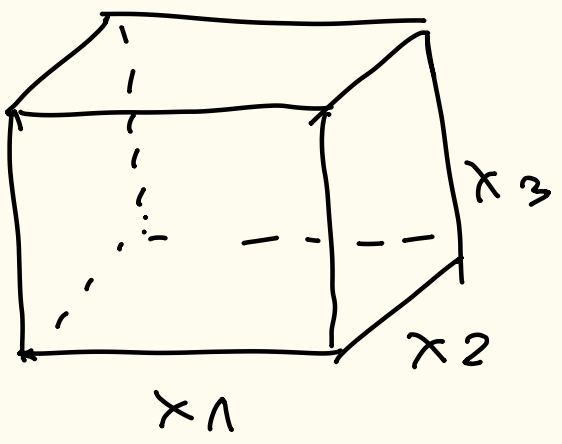
aritm. průměr

$$\frac{x_1 + \dots + x_n}{n} \geq e^{\frac{\ln x_1}{n}} \cdots e^{\frac{\ln x_n}{n}} =$$

$$= e^{\ln \sqrt[n]{x_1}} \cdots e^{\ln \sqrt[n]{x_n}} = \sqrt[n]{x_1} \cdots \sqrt[n]{x_n} = \sqrt[n]{x_1 \cdots x_n}$$

geom. průměr

$$\prod_{i=1}^{n} x_i^{d_i}$$



$$\min \; x_1x_2 + x_1x_3 + x_2x_3$$

$$\text{z.p. } x_1x_2x_3 = 1$$

$$x_i > 0 \quad i = 1,2,3$$

$$f(x_1, x_2, x_3) = 3\left(\frac{x_1x_2 + x_1x_3 + x_2x_3}{3}\right)$$

$$f(x_1,x_2,x_3) \geq 3\left((x_1x_2)(x_1x_3)(x_2x_3)\right)^{1/3}$$

$$= 3\left(\underbrace{x_1x_2x_3}_{1}\right)^{2/3} = 3$$

$x_i = 1 \Rightarrow f(x_1,x_2,x_3) = 3$ ✓

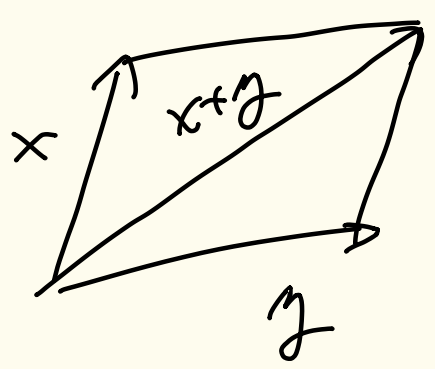
$f'(x) = 0$

$f(x) \leq f(y)$

$\forall y$!

$$\|x\|_2 = \sqrt[n]{x_1^2 + \dots + x_n^2}$$

$x \in \mathbb{R}^n$ $\quad \|0\|_2 = 0$



$$\|\alpha x\|_2 = |\alpha| \, \|x\|_2$$

$$\|x+y\|_2 \leq \|x\| + \|y\|$$

Manhattanská norma

$$\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$$

Euklidovská $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

Čebyševova $\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\} = \lim_{p\to\infty} \|x\|_p$

Norma je konvexní

$\alpha \in [0,1]$

$$\|(1-\alpha)x + \alpha y\| \leq \|(1-\alpha)x\| + \|\alpha y\| = (1-\alpha)\|x\| + \alpha\|y\|$$

je konvexní

$\{x \in \mathbb{R}^n \mid \|x\| = 1\}$

kružnice $\mathbb{R}^2$

sféra $\mathbb{R}^3$

$f^I(x) = ae^{ax}$

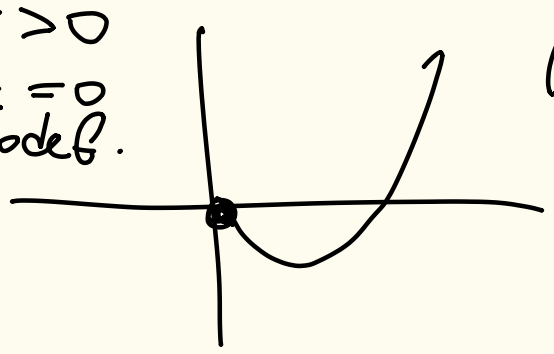
$f^{II}(x) = \underbrace{a^2 e^{ax}}_{\geq 0}$ je konvex

$f(x) = (x \log_2 x)^I = \log_2 x + \frac{x}{x \ln 2}$

$(x \log_2 x)^{II} = \frac{1}{\ln 2} > 0$

$x > 0$

$x = 0$ dodef.



konvex

$f(x) = \ln(e^{x_1} + \ldots + e^{x_n})$ → softmax

$f(x) = x^T A x$ A: PSD

je konvex

1) definice
2) approx taylor.
3) 2. derivace

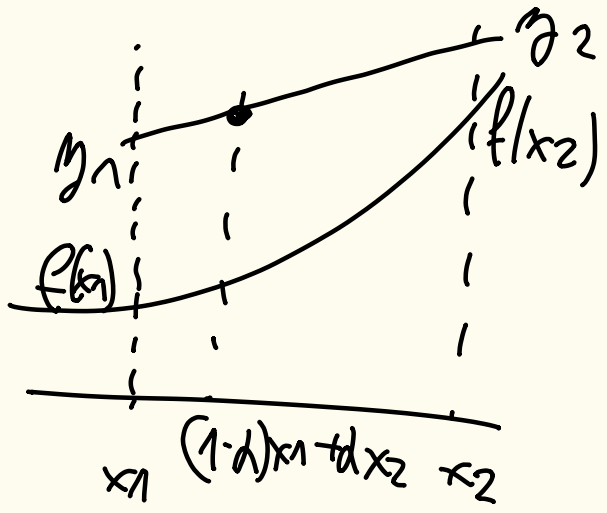
$f(x) = a^T x + b$

$f((1-\alpha)x + \alpha y) = (1-\alpha)a^T x + (1-\alpha)b + \alpha a^T y + \alpha b = (1-\alpha)f(x) + \alpha f(y)$

f konvexní $\Rightarrow$ epigraf je konvexní

$(x_1, y_1), (x_2, y_2)$

$y_i \geq f(x_i)$ $\quad \alpha \in [0,1]$



$\left((1-\alpha)x_1 + \alpha x_2, (1-\alpha)y_1 + \alpha y_2 \right)$

$f\left((1-\alpha)x_1 + \alpha x_2 \right) \leq (1-\alpha) f(x_1) + \alpha f(x_2) \leq (1-\alpha) y_1 + \alpha y_2$

konvexní

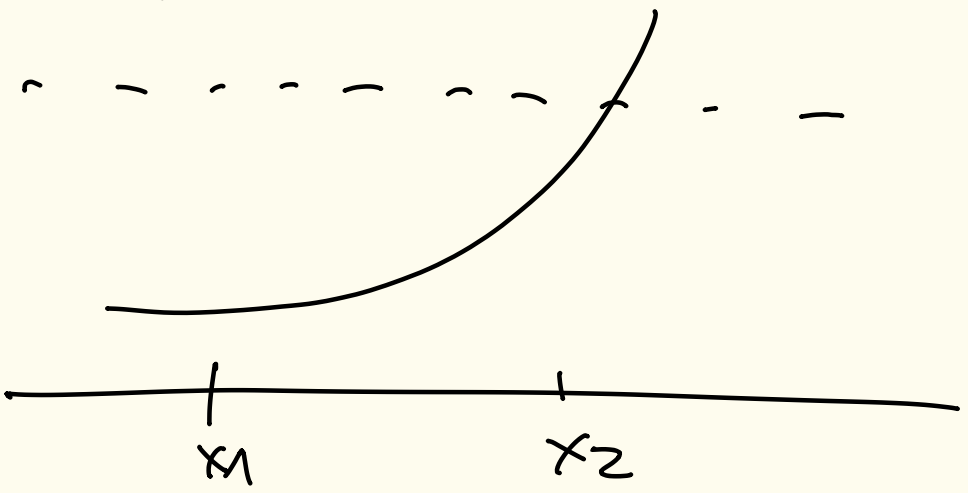
$y \in \mathbb{R} \quad \{ x \in \mathbb{R}^n \mid f(x) \leq y \}$

Vrstevnice (kontura)

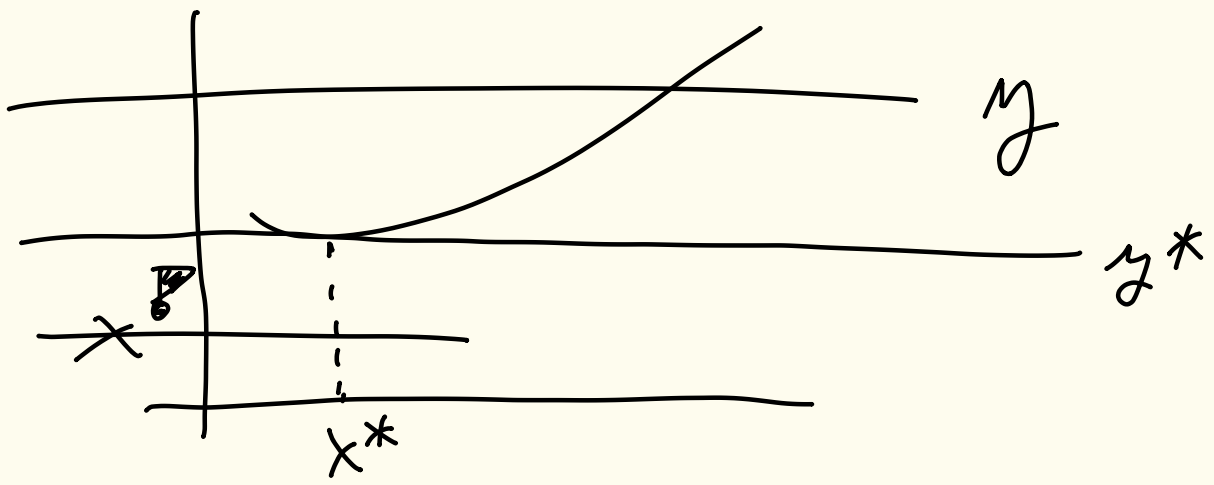
$x_1, x_2 \in \mathbb{R}^n, \quad \alpha \in [0,1]$

$f(x_1) \leq y$

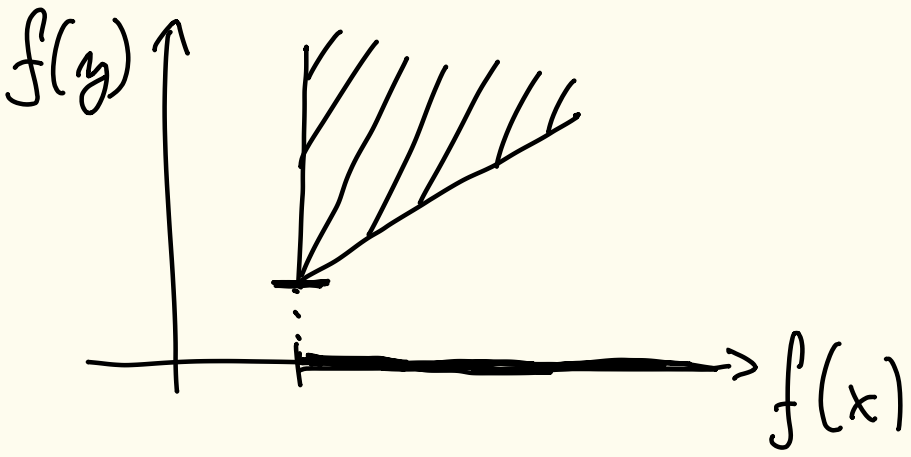
$f(x_2)$



$f((1-\alpha)x_1 + \alpha x_2) \leq \overbrace{(1-\alpha) f(x_1)}^{\leq y} + \underbrace{\alpha f(x_2)}_{\leq y} \leq (1-\alpha) y + \alpha y = y$

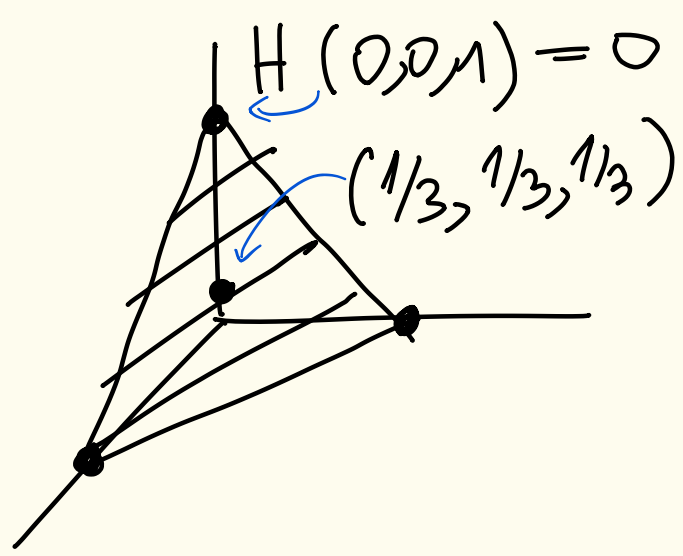


$f(x) \leq y$

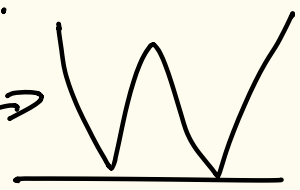


$$f(x) = \frac{\|x\|_1 + \|x_2\|}{2}$$

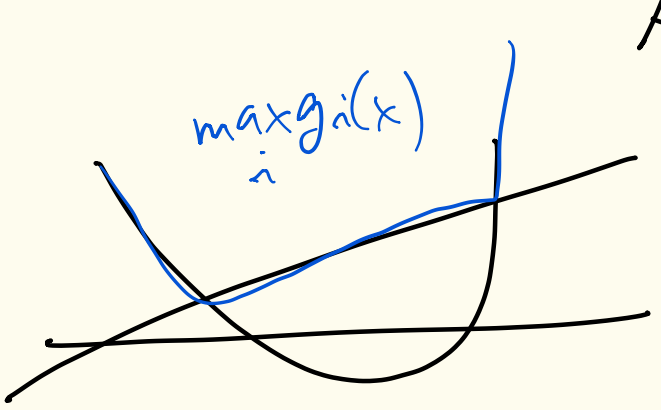
je konvex.



$h = g \circ f$ f konvex.
g konvex + rostoucí.

$h(x) = ||x|-1|$  horní konvex

$$\|x-y\| = \left\| \underbrace{[I\ -I]}_{A} \begin{pmatrix} x \\ y \end{pmatrix} \right\|$$



$\forall i \in I \quad g_i(x) \leq y$

$$\{(x,y) \in \mathbb{R}^{n+1} \mid f(x) \leq y\}$$