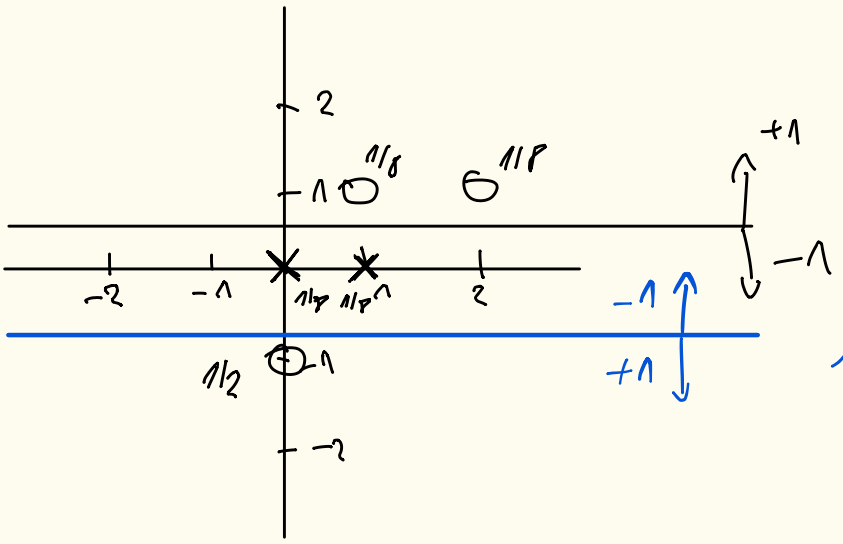
13.12.2024

$T = \{(x_i, y_i, h_i)\}: \{(0,-1,+1); (0,0,-1), (1,1,+1), (2,1,+1), (1,0,-1)\}$

T_1 T_2 T_3 T_4 T_5



$h_1 = \text{sign}(y_i - 1/2)$

$h_2 = -\text{sign}(y_i + 1/2)$

$D_1(i) = 1/N$

$\varepsilon_1 = \sum_i^N D_1(i) [\![h_1(x_i) \neq h_i]\!]$

$\varepsilon_1 = 1/5$

$\alpha_1 = 1/2 \log\left(\frac{1-\varepsilon_1}{\varepsilon_1}\right) = \frac{1}{2}\log\left(\frac{4/5}{1/5}\right) = 1/2 \log 4 = \log 2$

$D_2(1) = D_1(1) \cdot e^{\log 2}$

$D_2(2\text{-}5) = D_1(2\text{-}5) \cdot e^{-\log 2}$

$D_2(1) = 2/5$

$D_2(2\text{-}5) = 1/10$

$Z_1 = 2/5 + 4/10 = 4/5$

$D_2(1) = 1/2 \quad D_2(2\text{-}5) = 1/8$

$D_2(i) = \frac{D_1(i) \cdot e^{-\alpha_i \cdot k_i \cdot h(x_i)}}{Z_t}$

$\varepsilon_2 = \sum_i^N D_2(i) [\![h_2(x_i) \neq h_i]\!]$

$\varepsilon_2 = 2/8 = 1/4$

$\alpha_2 = 1/2 \log\left(\frac{1-\varepsilon_2}{\varepsilon_2}\right) = 1/2 \log\left(\frac{3/4}{1/4}\right) =$

$= 1/2 \log 3 = \log \sqrt{3}$

$D_3(3,4) = e^{\log\sqrt{3}} D_2(3,4)$

$D_3(1,2,5) = e^{-\log\sqrt{3}} \cdot D_2(1,2,5)$

D: 1 $\sqrt{3}/3 \cdot 1/2 = \sqrt{3}/6$
2 $\sqrt{3}/3 \cdot 1/8 = \sqrt{3}/24$
3 $\sqrt{3} \cdot 1/8 = \sqrt{3}/8$
4 $\sqrt{3} \cdot 1/8 = \sqrt{3}/8$
5 $\sqrt{3}/3 \cdot 1/8 = \sqrt{3}/24$

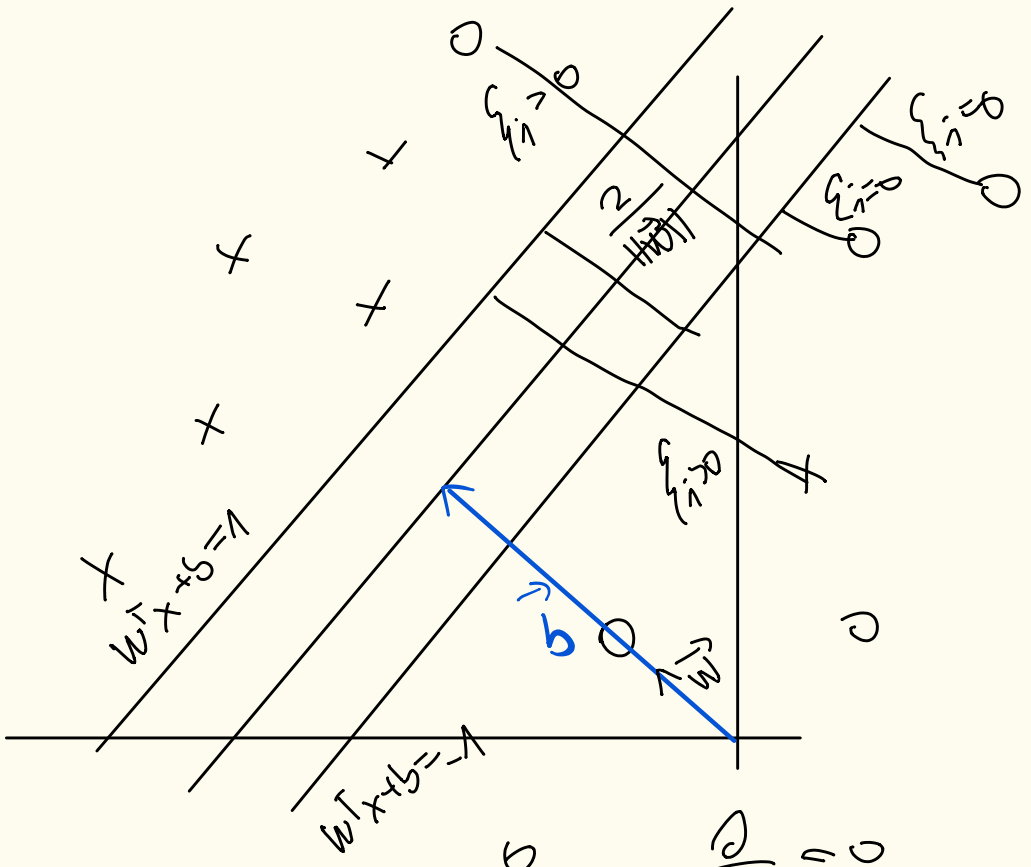
$\sum D_3 = \frac{\sqrt{3}}{2}$

znormalizowane:
1 : 2/6 = 1/3
2 : 1/12
3 : 1/4
4 : 1/4
5 : 1/12

$\Big\} \sum = 1$

výsledný klasifikátor: $H(x) = \sum \alpha_i h(x_i)$

$$H(x_i, y_i) = \text{sign}\left(\log 2 \cdot \text{sign}(y_i - 1/2) - \log\sqrt{3} \cdot \text{sign}(y_i + 1/2) \right)$$



$$\rightarrow \frac{\partial}{\partial \mu} = 0$$

$$P(x) = (2\pi)^{-n/2} |C|^{-n/2} e^{-1/2 (x-\mu)^T C^{-1} (x-\mu)}$$

$$L(\mu) = \prod_i e^{-1/2 \left((x-\mu)^T C^{-1} (x-\mu) \right)}$$

$$l(\mu) = -1/2 \sum_i (x-\mu)^T C^{-1} (x-\mu)$$

$x^T A x$

$$(x^T C^{-1} - \mu^T C^{-1})(x-\mu)$$

$$\Rightarrow (\mu^T C^{-1} x)^T = x^T C^{-1} \mu$$

$$l(\mu) = 1/2 \sum_i -x^T C^{-1} x + x^T C^{-1} \mu + \mu^T C^{-1} x + \mu^T C^{-1} \mu$$

$$\rightarrow \frac{\partial}{\partial \mu} = 0 \qquad 2 x^T C^{-1} \mu$$

$$l(\mu) = -\mu^T C^{-1} \sum_i x_i + \frac{N}{2} \mu^T C^{-1} \mu$$

$$\frac{\partial l}{\partial \mu} = -C^{-1}\sum_i x_i + NC^{-1}\mu = 0$$

$$\hat{\mu} = \frac{+C^{-1}\sum_i x_i}{NC^{-1}} = \frac{+\sum_i x_i}{N}$$

(logistická regrese

$$p(k|x) = \frac{1}{1+e^{-kwx}} \rightarrow p(1|x) \quad +1$$

$$p(1|x) \quad p(2|x)$$

$$\begin{matrix} \frac{1}{1+e^{wx}} & \frac{1}{1+e^{wx}} \\ \frac{1}{1+e^{wx}} & \end{matrix}^{-1}$$

$$\frac{p(1|x)}{p(2|x)} \rightarrow \begin{bmatrix} b \\ w \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ x \\ 1 \end{bmatrix}$$

$$\ln \frac{p(1|x)}{p(2|x)} = wx \quad /e \qquad T = \{(t_i, u_i)\} \; \forall i \in T$$

$$\frac{p(1|x)}{p(2|x)} = e^{ux}$$

$$p(1|x) = e^{ux} \cdot p(2|x) \qquad p(1|x) = 1 - p(2|x)$$

$$1 - p(2|x) = e^{ux} p(2|x)$$

$$1 = e^{ux} p(2|x) + p(2|x)$$

$$1 = p(2|x)\left(e^{ux} + 1\right)$$

$$p(2|x) = \frac{1}{1+e^{ux}}$$

$$1 - p(1|x) = \frac{1}{1+e^{wx}}$$

$$T = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \middle| \begin{pmatrix} 1 \\ -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

h_i $\quad -1 \quad -1 \quad 1$

$$1 - \frac{1}{1+e^{wx}} = p(1|x)$$

$$\frac{1+e^{wx}-1}{1+e^{wx}} = \frac{e^{wx}}{1+e^{wx}}$$

$$p(h|x) = \frac{1}{1+e^{-h w x}} \qquad h = \{-1, 1\}$$

$$E(w) = \sum_i \ln p(h x_i) = \sum_i \ln p(h|x_i) + \underbrace{\ln(x_i)}_{\text{const.}}$$

$$= \sum_i \ln p(h|x_i) = -\sum_i \ln\left(1+e^{-h_i w x_i}\right)$$

$$\frac{\partial E}{\partial w} = -\sum_i \frac{h_i x_i e^{-h_i w x_i}}{1+e^{-h_i w x_i}} = \qquad \frac{1}{1+e^{-h_i w x_i}} \cdot -w x_i e^{-h_i w x_i}$$

$$= -\sum \frac{1}{p(-h_i|x_i)} h_i x_i = -\sum \left(1 - p(h_i|x_i)\right) h_i x_i$$

$$x = -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{matrix} -1/2 \\ 1/2 \end{matrix}$$

$$T = \{(-1,-1), (-2,-1), (0,1), (2,1)\}$$

$$g(w) = -\sum \left(\begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \right) = -\begin{pmatrix} 1/2 \\ 2 \end{pmatrix} \frac{1}{1+e^{h_i w^T x_i}}$$

$$w_1 = \begin{bmatrix} 1/2 & 2 \end{bmatrix}^T \qquad \frac{-1\begin{bmatrix} 1 \\ -1 \end{bmatrix}}{1+e^{-1(0\ 0)\begin{bmatrix} 1 \\ -1 \end{bmatrix}}} = \frac{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}{2}$$

20.10.2023

(2) $u = \{1, 2\}$ $x \in [-2{,}5, 0]$

$$p(x|u=1) = \begin{cases} 0 & -2.5 \le x \le -\frac{\pi}{2} \\ \cos(x) & -\frac{\pi}{2} \le x \le 0 \end{cases}$$

$$p(x|u=2) = \begin{cases} 1/2 & -2.5 \le x < -1 \\ -\frac{x}{2} & -1 \le x \le 0 \end{cases}$$

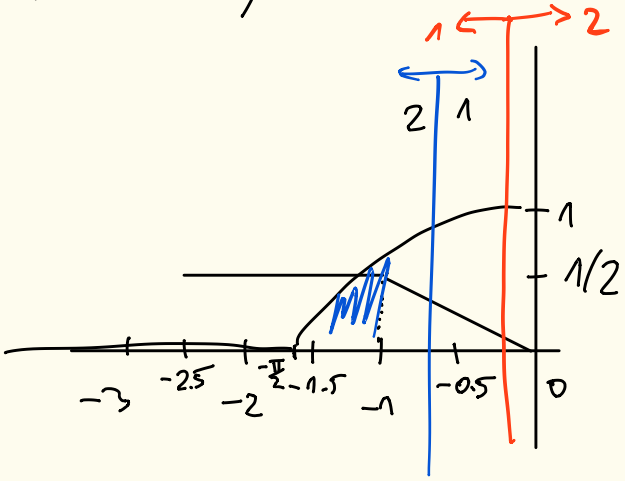
(a) Neumann pearson označí jednu třídu jako D (dangerous) a druhý jako N (Normal). Poté se řeší úloha

$$q^* = \arg\min_q \sum_{x: q(x) \neq N} p(x|N)$$

s.t. $$\sum_{x: q(x) \neq D} p(x|D) \le \bar{\varepsilon}_D$$

Tedy hledáme optimální strategii za podmínky maximální chyby na D.

1 je D, 2 je N $\varepsilon_D = 0.1$



D: $$\int_{-\pi/2}^{x} \cos x \, dx = [\sin x]_{-\pi/2}^{x} = \sin x + 1 = \varepsilon_D = 0.1$$

$\sin x = -0.9 = -1.12$ $x =$

N: eur: $0.12 \cdot 1/2 + 1/4 = 0.06 + 1/4 = 0.31$ $\varepsilon_N = 31\%$

B: $(2.5 - 1.12)\,1/2 = 0.69$

$$\int_x^0 \cos x\,dx = \left[\sin x\right]_x^0 = 0 - \sin x = \varepsilon_D = 0.1$$

$$x = -0.1$$

N: however: $\dfrac{-0.1 \cdot \left(\frac{-0.1}{2} - 1\right)}{2} = 0.05$

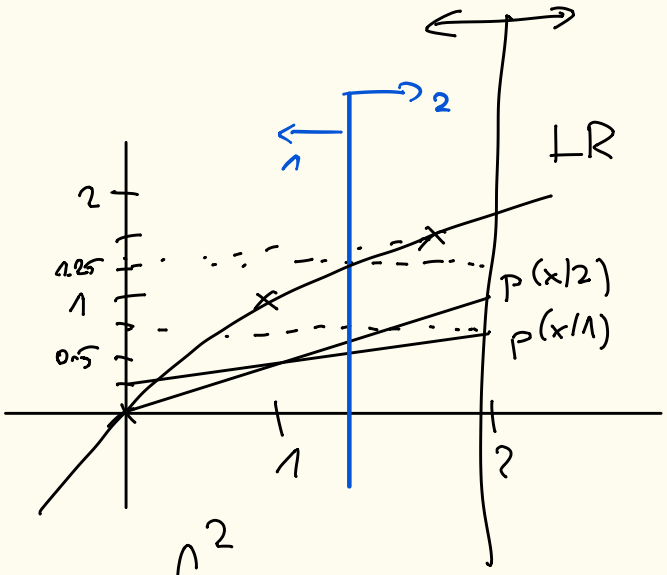
$$\varepsilon_N = 95\%$$

minimax:

$p(x|1) = 1/4\,x + 0.25$

$p(x|2) = 1/2\,x$

$x \in [0.0, 2.0]$



$$q(x) = \arg\min_q \max_{h \in k} \varepsilon(h)$$

$$\varepsilon_h = \sum_{x : q(x) = h} p(x|h)$$

$$\int_0^t p(x|2) = \int_t^2 p(x|1)$$

$$\frac{\frac{x}{2}}{\frac{x+1}{4}} = \frac{4x}{2x+2} = \frac{2x}{x+1}$$

$$\int_0^t 1/2\,x\,dx = \int_t^2 1/4\,x + 0.25$$

$$\left[\frac{x^2}{4}\right]_0^t = \left[\frac{x^2}{8} + 0.25x\right]_t^2$$

$$\frac{t^2}{4} = \frac{4}{8} + 1/2 - \frac{t^2}{8} - t/4$$

$$\frac{t^2}{4} = 1 - \frac{t^2}{8} - t/4$$

$$\frac{3t^2}{8} + \frac{t}{4} - 1 = 0 \quad / \cdot 8$$

$$3t^2 + 2t - 8 = 0 \quad \begin{cases} \cancel{-2} & \text{mimo int} \\ 4/3 & \varepsilon = \end{cases}$$

$p(x,k)$	$x=1$	$x=2$	$x=3$	$x=4$	Σ
$k=1$	A	0.12	0.06	0.36	
$k=2$	0.16	0.04	B	0.06	
Σ					1

$p(4=2) = 0.4$

$$0.4 = 0.16 + 0.04 + B + 0.06$$

$$0.14 = B$$

$$0.8 + 0.14 = \qquad A = 0.06$$

W	$d=1$	$d=2$
$k=1$	0	2
$k=2$	1	0

q

1	D1: $0.16 \cdot 1 = 0.16$	2
	D2: $0.06 \cdot 2 = 0.12$	
2	D1: $0.04 \cdot 1 = 0.04$	1
	D2: $0.12 \cdot 2 = 0.24$	
3	D1: $0.14 \cdot 1 = 0.14$	2
	D2: $0.06 \cdot 2 = 0.12$	
4	D1: $0.06 \cdot 1 = 0.06$	1
	D2: $0.36 \cdot 2 = 0.X$	

$$q(x) = \begin{cases} 2 \\ 1 \\ 2 \\ 1 \end{cases} \quad \text{pro}(x_1 \ldots x_4)$$

	h=A	h=B
x=1	0.06	0.2
x=2	0.12	0.12
x=3	0.42	0.08

$p(x=1) = 0.26$

$p(h|x)\,p(x) = p(h,x)$

vypočítejte $p(h|x=1)$

$p(A|x=1) = 0.06 / 0.26 = 0.23$

$p(B|x=1) = 0.2 / 0.26 = 0.77$

W	J=A	J=B
h=A	0	10
h=B	1	0

DA : 0.2 · 1 = A
DB : 0.06 · 10 =

DA : 0.12 · 1 = A
DB : 0.12 · 10 =

DA : 0.48 · 1 = A
DB : 0.42 · 10 =

$q = \{A, A, A\}$

$$f(x|\alpha) = \begin{cases} \alpha x_{min}^{\alpha} x^{-(\alpha+1)} & \text{pro } x \geq x_{min} \\ 0 & \text{pro } x < x_{min} \end{cases}$$

$x_{min} = 1 \quad \alpha > 1$

$$L(\alpha) = \prod_{x \in \mathcal{X}} \alpha x^{-(\alpha+1)}$$

$$l(\alpha) = \sum_{x \in \mathcal{X}} \log \alpha - (\alpha+1) \log x$$

$$= N \log \alpha - (\alpha+1) \sum \log x$$

$$\frac{\partial l}{\partial \alpha} = \frac{N}{\alpha} - \sum \log x = 0$$

$$\frac{N}{\alpha} - \sum \log x$$

$$\frac{N}{\alpha} = \sum \log x$$

$$\alpha = \frac{N}{\sum \log x} = \frac{5}{2{,}39} = 2{,}092$$

$N \cdot \sum x_i = 5 \cdot 10.9 = 54.5$

$$\hat{\alpha} = \frac{N}{\sum_i \log x_i - N \log x_{min}}$$

$$p(t) = \frac{1}{\theta} e^{-t/\theta} \qquad t \in [0, \infty]$$

$$\text{alive @ } T = \int_T^\infty \frac{1}{\theta} e^{-t/\theta} dt = \left| \frac{t}{\theta} = x \;\; \frac{1}{\theta} dt = dx \right| = \int_{T/\theta}^\infty e^{-x} dx =$$

$$= \left[-e^{-x} \right]_{T/\theta}^\infty = 0 + e^{-T/\theta}$$

$$\text{dead @ } T = 1 - e^{-T/\theta}$$

$$L(\theta) \prod_i^G e^{-T/\theta} \cdot \prod_i^F 1 - e^{-T/\theta}$$

$$l(\theta) = G \cdot \log\left(e^{-T/\theta}\right) + F \log\left(1 - e^{-T/\theta}\right)$$

$$G \cdot \frac{-T}{\theta}$$

$$\frac{\partial l}{\partial \theta} = \frac{GT}{\theta^2} - \frac{F \cdot e^{-T/\theta} \cdot \frac{T}{\theta^2}}{1 - e^{-T/\theta}} = 0$$

$$G = \frac{Fe^{-T/\theta}}{1-e^{-T/\theta}}$$

$$G - Ge^{-T/\theta} = Fe^{-T/\theta}$$

$$G = Fe^{-T/\theta} + Ge^{-T/\theta}$$

$$G = e^{-T/\theta}(F+G)$$

$$e^{-T/\theta} = \frac{G}{F+G}$$

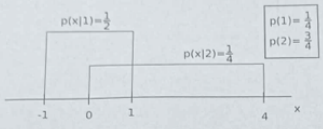
$$-T/\theta = \log\left(\frac{G}{F+G}\right)$$

$$\hat{\theta} = \frac{-T}{\log\left(\frac{G}{F+G}\right)}$$

RPZ Test 16. ledna 2025

Jméno:

1. (12b) Optimální klasifikátor a 1-NN klasifikátor.
Mějme dvě třídy označené 1 a 2. Podmíněné a apriorní pravděpodobnosti jsou:



$p(1) = \frac{1}{4}$, $p(2) = \frac{3}{4}$

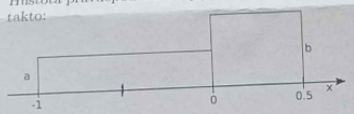
$$p(x|1) = \frac{1}{2} \text{ pro } x \in [-1, 1],$$
$$p(x|2) = \frac{1}{4} \text{ pro } x \in [0, 4],$$
$$p(1) = \frac{1}{4},$$
$$p(2) = \frac{3}{4}.$$

(a) (5b) Najděte optimální klasifikátor $q^*(x)$, tedy takový, který má na zadaných distribucích minimální chybu klasifikace. Zapište ho přehlednou formou.

(b) (1b) Spočtěte chybu ϵ^* tohoto klasifikátoru.

(c) (6b) Uvažte klasifikátor 1-NN (klasifikace podle nejbližšího souseda), přičemž předpokládejte asymptotický případ, kdy množství trénovacích i testovacích dat roste k nekonečnu. Spočítejte chybu ϵ^{NN} tohoto klasifikátoru.

2. (13b) Odhad parametrů.
Hustota pravděpodobnosti $p(x)$ na intervalu $x \in [-1, 0.5]$ je pomocí parametrů a a b definována takto:



$$p(x) = a \text{ pro } x \in [-1, 0],$$
$$p(x) = b \text{ pro } x \in (0, 0.5],$$

(a) (1b) Napište podmínku pro a, b plynoucí z toho, že $p(x)$ má být hustotou pravděpodobnosti.

Mějme trénovací data $\mathcal{X} = \{x_1, x_2, ..., x_K, x_{K+1}, x_{K+2}, ..., x_{K+L}\}$ taková, že
$\forall i \in \{1, 2, ..., K\} : x_i \in [-1, 0]$,
$\forall j \in \{1, 2, ..., L\} : x_{K+j} \in (0, 0.5]$.

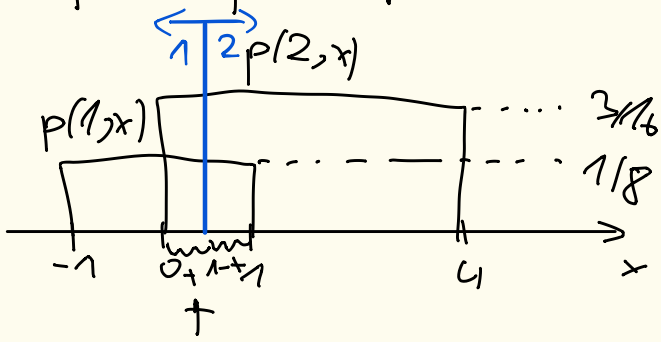
(b) (2b) Pro tato trénovací data napište vzorec pro věrohodnost (likelihood).

(c) (4b) Odvoďte vztah pro výpočet parametrů a a b pro tato data metodou maximální věrohodnosti (ML). Potom spočítejte ML odhad pro a a b pro konkrétní trénovací množiny $\mathcal{X}_A = \{-0.1, 0.2\}$ a $\mathcal{X}_B = \{-0.8, -0.5, -0.3, 0.3\}$.

(d) (6b) Nechť navíc apriorní pravděpodobnost pro parametr a je $P(a) = 4a^3, 0 \leq a \leq 1$. Odvoďte vztah pro výpočet parametrů a a b metodou maximalizace aposteriorní pravděpodobnosti (MAP). Potom spočítejte MAP odhad pro a a b pro konkrétní trénovací množiny $\mathcal{X}_A$ a $\mathcal{X}_B$ jako v předchozím bodě.

(1) $p(1,x) = p(x|1) \cdot p(1) = 1/2 \cdot 1/4 = 1/8$

$p(2,x) = p(x|2) \cdot p(2) = 1/4 \cdot 3/4 = 3/16$



$t \cdot 3/16 = (1-t) 1/8$

$3/16\, t = 1/8 - 1/8\, t$

$5/16\, t = 1/8$

$5t = 2$

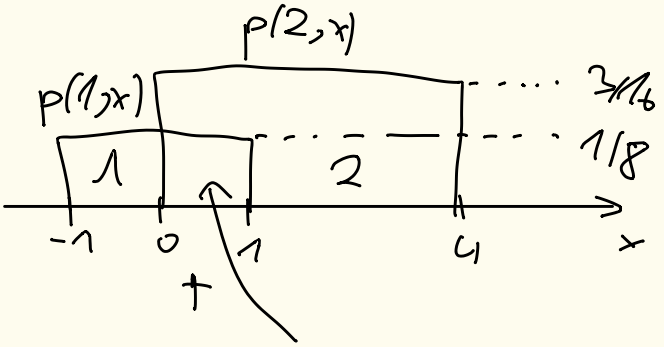
$t = 2/5$

$\varepsilon_2 = 3/16 \cdot 2/5 = 3/40$

$\varepsilon^* = 2\varepsilon_2 = 3/20$

$p(1|x) = \frac{p(x|1)\,p(1)}{p(x)} = \frac{1/2 \cdot 1/4}{3/16 + 3/16} = \frac{1/8}{5/16} = \frac{16}{5 \cdot 8} = \frac{2}{5}$

(c) 1-NN



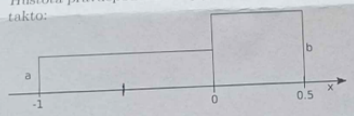
$$q = \begin{cases} 1 & x \in \langle -1, 0 \rangle, \text{ s pst } 2/5 \\ 2 & x \in \langle 1, 4 \rangle, \text{ s pst } 3/5 \end{cases}$$

$p(2|x) = \frac{p(x|2)\,p(2)}{p(x)} = \frac{1/4 \cdot 3/4}{5/16} = \frac{3/16}{5/16} = 3/5$

$\varepsilon^{NN} = \frac{2}{5} \cdot \frac{3}{5} + \frac{3}{5} \cdot \frac{2}{5}$

$(1/8 + 3/16) \cdot (3/16 + 1/8) \quad 3/16 + 1/8 = 5/16$

2. (13b) Odhad parametrů.
Hustota pravděpodobnosti $p(x)$ na intervalu $x \in [-1, 0.5]$ je pomocí parametrů a a b definována takto:



$$p(x) = a \quad \text{pro } x \in [-1, 0],$$
$$p(x) = b \quad \text{pro } x \in (0, 0.5],$$

(a) (1b) Napište podmínku pro a, b plynoucí z toho, že $p(x)$ má být hustotou pravděpodobnosti.

Mějme trénovací data $\mathcal{X} = \{x_1, x_2, ..., x_K, x_{K+1}, x_{K+2}, ..., x_{K+L}\}$ taková, že
$\forall i \in \{1, 2, ..., K\} : x_i \in [-1, 0]$,
$\forall j \in \{1, 2, ..., L\} : x_{K+j} \in (0, 0.5]$.

(b) (2b) Pro tato trénovací data napište vzorec pro věrohodnost (likelihood).

(c) (4b) Odvoďte vztah pro výpočet parametrů a a b pro tato data metodou maximální věrohodnosti (ML). Potom spočítejte ML odhad pro a a b pro konkrétní trénovací množiny $\mathcal{X}_A = \{-0.1, 0.2\}$ a $\mathcal{X}_B = \{-0.8, -0.5, -0.3, 0.3\}$.

(d) (6b) Nechť navíc apriorní pravděpodobnost pro parametr a je $P(a) = 4a^3, 0 \leq a \leq 1$. Odvoďte vztah pro výpočet parametrů a a b metodou maximalizace aposteriorní pravděpodobnosti (MAP). Potom spočítejte MAP odhad pro a a b pro konkrétní trénovací množiny $\mathcal{X}_A$ a $\mathcal{X}_B$ jako v předchozím bodě.

$L(\theta) \prod_K a \cdot \prod_L b$

$l(\theta) \log a^k + \log b^L$

$a + b/2 = 1$

$a = 1 - b/2$

$l(\theta) \quad k \log a + L \log b$

$l(b) \quad k \log(1 - b/2) + L \log b$

$\frac{\partial l}{\partial b} = -\frac{1}{2}\left(\frac{k}{1-b/2}\right) + \frac{L}{b} = 0$

$\frac{L}{b} = \frac{1}{2}\left(\frac{k}{1-b/2}\right) \quad / \cdot 2$

$\frac{2L}{b} = \frac{k}{1-b/2}$

$2L(1 - b/2) = kb$

$2L - \frac{2Lb}{2}$

$2L = kb + \frac{2Lb}{2}$

$4L = 2kb + 2Lb$

$4L = 2b(k+L)$

$2L = b(k+L)$

$b = \frac{2L}{k+L}$

$X_A \quad L = 1$

$X_B \quad k = 3$

$b = 1$

$a = 1 - \frac{L}{(k+L)}$

$b = \frac{2L}{k+L}$

pro X_A: $a = 1 - \frac{1}{2} = 1/2$

$b = 1$

pro X_B: $a = 1 - \frac{1}{4} = 3/4$

$b = \frac{2}{4} = 1/2$