Úkol 6

Úloha 1

$$P(x) = (x+\cos x)^{x+\cos x} = e^{(x+\cos x)\ln(x+\cos x)}, \quad x>0$$

$$P'(x) = e^{(x+\cos x)\ln(x+\cos x)}\left((1-\sin x)\ln(x+\cos x) + (x+\cos x)\cdot\frac{1-\sin x}{x+\cos x}\right)$$

$$= (x+\cos x)^{x+\cos x}\Big((1-\sin x)\big(1+\ln(x+\cos x)\big)\Big), \quad x>0$$

$$K(x) = \frac{e^{e^{x+\cos x}}}{e^x+\sin x}, \quad x>0$$

$$K(x) = \frac{e^{e^{x+\cos x}}\cdot e^{x+\cos x}(1-\sin x)(e^x+\sin x) - e^{e^{x+\cos x}}(e^x+\cos x)}{(e^x+\sin x)^2}, \quad x>0$$

$$V(x) = \frac{a^{x^{x+1}}-1}{x^{x+1}} = \frac{a^{e^{(x+1)\ln x}}-1}{e^{(x+1)\ln x}}, \quad x>0 \quad (a \geq 0)$$

Pro $a>0$:

$$V'(x) = \frac{a^{x^{x+1}}\cdot\ln a\cdot\cancel{x^{x+1}}\cdot\left(\ln x+\frac{x+1}{x}\right)\cdot x^{x+1} - \left(a^{x^{x+1}}-1\right)\cancel{x^{x+1}}\cdot\left(\ln x+\frac{x+1}{x}\right)}{(x^{x+1})^{\cancel{2}}}$$

$$= \frac{\left(a^{x^{x+1}}\left(\ln a\cdot x^{x+1}-1\right)+1\right)\left(\ln x+\frac{x+1}{x}\right)}{x^{x+1}}, \quad x>0$$

Pro $a=0$:

$$V(x) = -\frac{1}{x^{x+1}} = -e^{-(x+1)\ln x}$$

$$V'(x) = -e^{-(x+1)\ln x}\cdot\left(-\ln x-\frac{x+1}{x}\right) = \frac{\ln x+\frac{x+1}{x}}{x^{x+1}}, \quad x>0.$$

Bonus $(a>0)$:

$$\lim_{x\to 0+} V(x) = \lim_{x\to 0+} \frac{a^{x^{x+1}}-1}{x^{x+1}} = \lim_{x\to 0+} \frac{e^{x^{x+1}\cdot \ln a}-1}{x^{x+1}\ln a}\cdot \ln a \overset{(1)}{=} \ln a$$

(1): $f(x) = x^{x+1}\ln a = e^{(x+1)\ln x}\cdot \ln a$

$$\lim_{x\to 0+} f(x) = \ln a \cdot \lim_{x\to 0+} e^{\underbrace{\underbrace{(x+1)}_{\to 1}\underbrace{\ln x}_{\to -\infty}}_{\xrightarrow[AL]{} -\infty}} \overset{(2)}{=} 0$$

$f(x) \neq 0$ pro $\forall x \in (0,\infty)$

$g(y) = \frac{e^y-1}{y} \xrightarrow{y\to 0+} 1$

$\overset{LSF(ii)}{\Longrightarrow}$ (1)

(2): $f(x) = (x+1)\ln x \xrightarrow[AL]{x\to 0+} -\infty$

$f(x) \neq -\infty \quad \forall x\in(0,\infty)$

$g(y) = e^y \xrightarrow{y\to -\infty} 0$

$\overset{LSF(ii)}{\Longrightarrow}$ (2)

Úloha 2

$$f(x) = \left(\frac{x+1}{3^x}\right)^{\operatorname{arctg}\left(\frac{1}{x}\right)} = e^{\operatorname{arctg}\left(\frac{1}{x}\right)\ln\left(\frac{x+1}{3^x}\right)}$$

$$= e^{\operatorname{arctg}\left(\frac{1}{x}\right)\left(\ln(x+1)-\ln 3^x\right)}$$

$$= e^{\operatorname{arctg}\left(\frac{1}{x}\right)\ln(x+1)}\cdot e^{-\operatorname{arctg}\left(\frac{1}{x}\right)\cdot x\ln 3} \qquad (*)$$

Přímé použití l'H – potrvá dlouho dopočítat:

$$\lim_{x\to\infty} \underbrace{\operatorname{arctg}\left(\frac{1}{x}\right)}_{\to 0}\underbrace{\ln(x+1)}_{\to\infty} = \lim_{x\to\infty}\frac{\operatorname{arctg}\left(\frac{1}{x}\right)}{\frac{1}{\ln(x+1)}} \overset{l'H \text{ "}\frac{0}{0}\text{"}}{=} \dots$$

Rychlejší způsob:

$$\lim_{x\to\infty} \operatorname{arctg}\left(\frac{1}{x}\right)\ln(x+1) = \lim_{x\to\infty} \underbrace{\frac{\operatorname{arctg}\left(\frac{1}{x}\right)}{\frac{1}{x}}}_{\substack{\longrightarrow 1 \\ (3)}} \cdot \underbrace{\frac{\ln(x+1)}{x}}_{\substack{\longrightarrow 0 \\ (4)}} = 0 \qquad (5)$$

(3): známá limita $\frac{\operatorname{arctg} y}{y} \xrightarrow{y\to 0} 1$ & LSF nebo:

$$\lim_{x\to\infty} \frac{\operatorname{arctg}\left(\frac{1}{x}\right)}{\frac{1}{x}} \overset{\text{l'H}\,\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x^2}} \cdot \cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}} \overset{AL}{=} 1$$

(4): $$\lim_{x\to\infty} \frac{\ln(x+1)}{x} \overset{\text{l'H}\,\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to\infty} \frac{\frac{1}{x+1}}{1} = 0$$

ještě potřeba:

$$\lim_{x\to\infty} x \operatorname{arctg}\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\operatorname{arctg}\left(\frac{1}{x}\right)}{\frac{1}{x}} = 1 \text{ jako výše.} \qquad (6)$$

Zpět k (*):

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \underbrace{e^{\overbrace{\operatorname{arctg}\left(\frac{1}{x}\right)\ln(x+1)}^{\longrightarrow 0 \text{ z } (5)}}}_{\longrightarrow 1 \text{ (LSF(i))}} \cdot \underbrace{e^{-\overbrace{\operatorname{arctg}\left(\frac{1}{x}\right)\cdot x}^{\longrightarrow 1 \text{ z } (6)} \ln 3}}_{\longrightarrow e^{-\ln 3} = \frac{1}{3} \text{ (LSF(i))}}$$

$$\overset{AL}{\Longrightarrow} \quad \boxed{\lim_{x\to\infty} f(x) = \frac{1}{3}}$$

Úloha 3

$$\lim_{x\to\infty} \frac{f(x)}{a^x} = b \qquad (7)$$

$$\lim_{x\to\infty} x\,g(x) = c \qquad (8)$$

$a, b, c \in \mathbb{R},\ a > 0.$

$$\lim_{x\to\infty} f(x)^{g(x)} = \lim_{x\to\infty} \exp\Big(g(x) \ln f(x)\Big)$$

$$= \lim_{x\to\infty} \exp\left(g(x) \ln\left(\frac{f(x)}{a^x}\cdot a^x\right)\right)$$

$$= \lim_{x\to\infty} \exp\left(g(x)\left(\ln\left(\frac{f(x)}{a^x}\right) + \ln a^x\right)\right)$$

$$= \lim_{x\to\infty} \left(\underbrace{\exp\left(\underbrace{g(x)\cdot x}_{\substack{\to c \\ (z\ (8))}} \cdot \underbrace{\frac{\overbrace{\ln\left(\frac{f(x)}{a^x}\right)}^{\to b\ (ze\ (7))}}{x}}_{\substack{\downarrow \text{AL } (b\in\mathbb{R}) \\ 0}} \right)}_{\to 1\ \ (\text{AL } \&\ \text{LSF}(i))} \cdot \underbrace{\exp\Big(\underbrace{g(x)\cdot x}_{\to c} \ln a\Big)}_{\substack{\to e^{c\ln a} = a^c \\ (\text{LSF}(i))}} \right)$$

$$\overset{\text{AL}}{=} a^c.$$

Výsledek:

$$\boxed{\lim_{x\to\infty} f(x)^{g(x)} = a^c.}$$