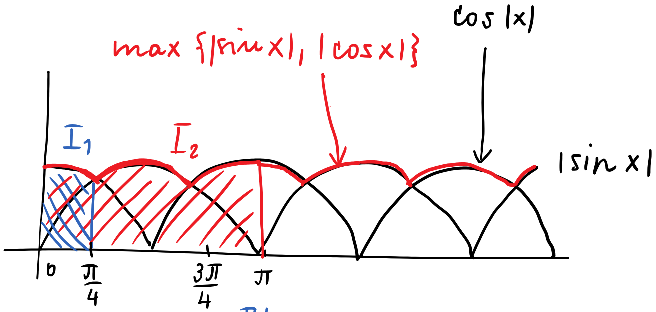
Úkol 10

Úloha 1

$$\int_0^{\pi} \underbrace{\max\{|\sin x|, |\cos x|\}}_{f(x)}\, dx$$



$$= 4\int_0^{\pi/4} \cos x \, dx$$

$$I_1 = \int_0^{\pi/4} f(x)\,dx$$

$$I_2 = \int_0^{\pi} f(x)\,dx = 4A_1$$

$$= 4\left[\sin x\right]_0^{\frac{\pi}{4}}$$

$$= 4\cdot\left(\frac{\sqrt{2}}{2} - 0\right) = \boxed{2\sqrt{2}}$$

alternativa bez geometrického pozorování:

$$\int_0^{\pi} \max\{|\sin x|, |\cos x|\}\,dx = \int_0^{\pi/4} |\cos x|\,dx + \int_{\pi/4}^{3\pi/4} |\sin x|\,dx + \int_{\frac{3\pi}{4}}^{\pi} |\cos x|\,dx$$

$$= \int_0^{\pi/4} \cos x \, dx + \int_{\pi/4}^{3\pi/4} \sin x \, dx + \int_{\frac{3\pi}{4}}^{\pi} \cos x \, dx = \cdots$$

Úloha 2

část (a):

Pozorování: integrand spojitý na $(0, \pi/4]$.

$$\int_0^{\pi/4} \frac{dx}{\sin x \cos x} = \int_0^{\pi/4} \frac{\cos x \, dx}{\sin x \cos^2 x} = \int_0^{\pi/4} \frac{\cos x\, dx}{\sin x(1-\sin^2 x)}$$

$$= \int_0^{\frac{\sqrt{2}}{2}} \frac{dy}{y(1-y^2)} = \int_0^{\frac{\sqrt{2}}{2}} \left(\frac{1}{y} - \frac{1}{2(y-1)} - \frac{1}{2(y+1)}\right) dy$$

$$= \left[\ln|y|\right]_0^{\frac{\sqrt{2}}{2}} - \frac{1}{2}\left[\ln|y-1|\right]_0^{\frac{\sqrt{2}}{2}} - \frac{1}{2}\left[\ln|y+1|\right]_0^{\frac{\sqrt{2}}{2}}$$

$$= \lim_{y\to 0+} \ln|y| + \ln\frac{\sqrt{2}}{2} - \frac{1}{2}\ln\left|\frac{\sqrt{2}}{2} - 1\right| - \frac{1}{2}\ln\left(\frac{\sqrt{2}}{2}+1\right)$$

$$= \boxed{\infty.}$$

Jiná možnost:

$$\int_0^{\pi/4} \frac{dx}{\sin x \cos x} = \int_0^{\pi/4} \frac{\sin^2 x}{\sin x \cos x}\,dx + \int_0^{\pi/4} \frac{\cos^2 x}{\sin x \cos x}\,dx$$

$$= \int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx + \int_0^{\pi/4} \frac{\cos x}{\sin x}\,dx \underset{\text{subst.}}{=} -\int_1^{\frac{\sqrt{2}}{2}} \frac{dy}{y} + \int_0^{\frac{\sqrt{2}}{2}} \frac{dy}{y} = \infty.$$

část (b):

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\sin x \cos x}$$ neexistuje, protože:

$$\int_0^{\pi/4} \frac{dx}{\sin x \cos x} = \infty \quad \& \quad \int_{-\frac{\pi}{4}}^{0} \frac{dx}{\sin x \cos x} \overset{(*)}{=} -\infty$$

(*) je vidět z lichosti integrandu, lze ověřit takto:

$$\int_{-\frac{\pi}{4}}^{0} \frac{dx}{\sin x \cos x} = \left| \begin{matrix} z = -x \\ dz = -dx \end{matrix} \right| = -\int_{\frac{\pi}{4}}^{0} \frac{dz}{\sin(-z)\cos(-z)} = -\int_0^{\frac{\pi}{4}} \frac{dz}{\sin z \cos z} = -\infty$$

úloha 3

Pozorování: integrand spojitý na $(0,1]$.

Řešení per-partes:

$$\int_0^1 |\ln x|\,dx = -\int_0^1 \ln x\,dx = \left| \begin{matrix} u'(x) = 1 \\ v(x) = \ln x \end{matrix} \right| \overset{P-P}{=} -[x \ln x]_0^1 + \int_0^1 dx$$

$$= 1 - \lim_{x\to 0+} x \ln x = 1 - \lim_{x \to 0+} \frac{\ln x}{\frac{1}{x}}$$

$$\overset{l'H,\,\text{"}\frac{\infty}{\infty}\text{"}}{=} 1 - \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \boxed{1.}$$

Jiné řešení – substituce:

$$\int_0^1 |\ln x|\,dx = \left| \begin{matrix} y = -\ln x \\ e^{-y} = x \\ -e^{-y}\,dy = dx \end{matrix} \right| = \int_0^\infty y \cdot e^{-y}\,dy$$

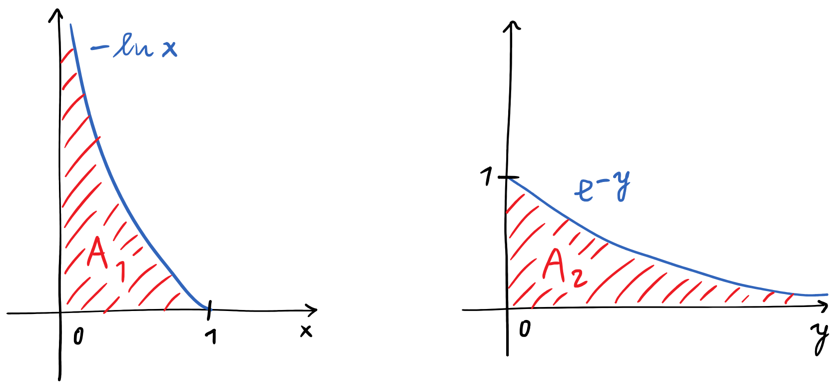
$$\overset{P-P}{=} \left[-ye^{-y}\right]_0^\infty + \int_0^\infty e^{-y}\,dy = -\left[ye^{-y}\right]_0^\infty - \left[e^{-y}\right]_0^\infty$$

$$= -\underbrace{\lim_{y\to\infty} ye^{-y}}_{=0\ (**)} + 0 + \underbrace{\lim_{y\to\infty} (e^{-y})}_{=0} + 1 = 1.$$

$$(**):\ \lim_{y\to\infty} \frac{y}{e^y} \overset{l'H\ ,,\frac{\infty}{\infty}"}{=} \lim_{y\to\infty} \frac{1}{e^y} = 0$$

Jiné řešení – geometrické pozorování:

exp a ln jsou vzájemně inverzní, tedy jejich grafy jsou symetrické dle osy 1. kvadrantu, a tudíž tyto plochy jsou stejné $(A_1 = A_2)$:



$$A_1 = \int_0^1 (-\ln x)\,dx = \int_0^1 |\ln x|\,dx \qquad A_2 = \int_0^\infty e^{-y}\,dy = \left[-e^{-y}\right]_0^\infty$$

$$= 1 - \lim_{y\to\infty} e^{-y} = 1$$