23.1.2020

1) Body na elipsoidu $3x^2+2y^2+z^2=1$
ve kterých je tečná rovina $\parallel \rho: 3x-y+3z=2$

$\Phi(x,y) = 3x^2+2y^2+z^2-1$

$\nabla\Phi(x,y) = (6x, 4y, 2z)$

$\vec{n}_\rho = (3,-1,3)$ chceme: $6x = 3\lambda \to x = \frac{\lambda}{2}$
$4y = -\lambda \to y = -\frac{\lambda}{4}$
$2z = 3\lambda \to z = \frac{3}{2}\lambda$

tedy $P_1, P_2 \in \operatorname{span}\left(\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{4}\\ \frac{3}{2}\end{pmatrix}\right)$

$$3\left(\frac{\lambda}{2}\right)^2 + 2\left(-\frac{\lambda}{4}\right)^2 + \left(\frac{3}{2}\lambda\right)^2 - 1 = 0$$

$$\frac{3\lambda^2}{4} + \frac{2\lambda^2}{16} + \frac{9\lambda^2}{4} - 1 = 0$$

$$\frac{12\lambda^2}{16} + \frac{2\lambda^2}{16} + \frac{36\lambda^2}{16} - 1 = 0$$

$$\frac{50\lambda^2}{16} - 1 = 0$$

$$\lambda_{1,2} = \pm\frac{4}{\sqrt{50}} = \pm\frac{4}{5\sqrt{2}}$$

$P_1 = \left(\frac{2}{\sqrt{12}}, -\frac{1}{\sqrt{12}}, \frac{6}{\sqrt{12}}\right)$

$P_2 = \left(\frac{-2}{\sqrt{12}}, \frac{1}{\sqrt{12}}, \frac{-6}{\sqrt{12}}\right)$

2) lok. extrémy $f(x,y) = x^3y^2 - 3xy - 1$

+ Taylor 2. řádu

$\nabla f(x,y) = (3x^2 - 3y, 2y - 3x)$

$\begin{cases} 3x^2 - 3y = 0 \\ 2y - 3x = 0 \rightarrow y = \frac{3}{2}x \end{cases}$

$3x^2 - \frac{9}{2}x = 0$

$x\left(3x - \frac{9}{2}\right) = 0 \rightarrow x_1 = 0 \vee x_2 = \frac{3}{2}$

$\downarrow \qquad \downarrow$

$y_1 = 0 \qquad y_2 = \frac{9}{4}$

kritické body: $(0,0)$

$\left(\frac{3}{2}, \frac{9}{4}\right)$

$H_f(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 2 \end{pmatrix}$

v $\left(\frac{3}{2}, \frac{9}{4}\right)$:

$$H_f\left(\frac{3}{2}, \frac{9}{4}\right) = \begin{pmatrix} 9 & -3 \\ -3 & 2 \end{pmatrix}, \quad \Delta_1 = 9, \ \Delta_2 = 18 - 9 = 9$$

pozitivně definitní

$$f\left(\frac{3}{2}, \frac{9}{4}\right) = \frac{27}{8} + \frac{81}{16} - \frac{27}{8} - 1 = \frac{65}{16}$$

zde je lokální minimum

$$T_2(p_1)(h) = f(p_1) + (h)^T H_f(p_1)(h) =$$

$$= \frac{65}{16} + \frac{1}{2}\begin{pmatrix} h_1 & h_2 \end{pmatrix}\begin{pmatrix} 9 & -3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} =$$

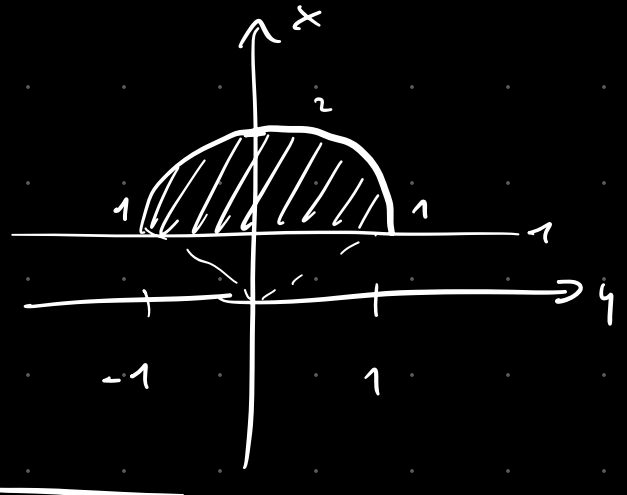
$$= \frac{65}{16} + \frac{1}{2}\begin{pmatrix} 9h_1 - 3h_2 \\ 2h_2 - 3h_1 \end{pmatrix}^T \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \frac{65}{16} + \frac{1}{2}(9h_1 - 3h_2)\cdot h_1 +$$

$$+ \frac{1}{2}(2h_2 - 3h_1)\cdot h_2 =$$

$$= \frac{65}{16} + \frac{9}{2}h_1^2 - \frac{3}{2}h_1h_2 + h_2^2 - \frac{3}{2}h_1h_2 =$$

$$= \frac{65}{16} - 3h_1h_2 + \frac{9}{2}h_1^2 + h_2^2$$

3)
$$\int_{-1}^{1}\int_{1}^{1+\sqrt{1-y^2}} f\,dx\,dy$$

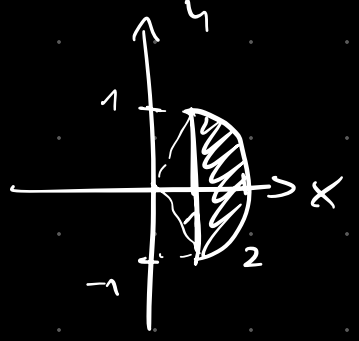


$x \le 1 + \sqrt{1-y^2}$

$(x-1)^2 \le 1 - y^2$

$(x-1)^2 + y^2 \le 1 \rightarrow y \le \sqrt{1-(x-1)^2}$

a) $\int_1^2 \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} f \, dy \, dx$



b) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{\frac{1}{\cos\varphi}}^{2\cos\varphi} f \cdot r \, dr \, d\varphi$

$x = r\cos\varphi$

$y = r\sin\varphi$

$\varphi \in \langle \frac{\pi}{4}, \frac{3\pi}{4} \rangle$

$(r\cos\varphi - 1)^2 + (r\sin\varphi)^2 \leq 1$

$r^2\cos^2\varphi - 2r\cos\varphi + 1 + r^2\sin^2\varphi \leq 1$

$r^2 - 2r\cos\varphi \leq 0$

$r^2 \leq 2r\cos\varphi$

$r \leq 2\cos\varphi$

$x = 1 \rightarrow r\cos\varphi = 1$

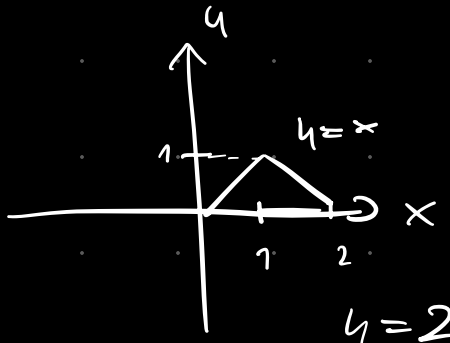
$r = \frac{1}{\cos\varphi}$

4) Green:

$$\int_C \vec{F} \, d\vec{S} = \iint_T \left(\frac{dF_2}{dx} - \frac{dF_1}{dy} \right) dS$$

$\vec{F}(x,y) = (y^3 + x, xy^2)$, $\frac{dF_2}{dx} - \frac{dF_1}{dy} = y^2 - 3y^2 = -2y^2$

Kružnice: kružnice $\triangle$ s vrcholy $(0,0)$, $(1,1)$, $(2,0)$, kladně orá



$$\int_0^1\int_y^{2-y} -2y^2\,dx\,dy =$$

$y=2-x$

$$= \int_0^1 \left[-2xy^2\right]_y^{2-y} dy =$$

$y=2-x$

$x=2-y$

$$= \int_0^1 \left(-2(2-y)y^2-(-2y^3)\right)dy =$$

$$= \int_0^1 \left(-4y^2+2y^3+2y^3\right)dy = \left[-\frac{4}{3}y^3+\frac{y^4}{4}+\frac{y^4}{4}\right]_0^1 = -\frac{4}{3}+\frac{1}{4}+\frac{1}{4} =$$

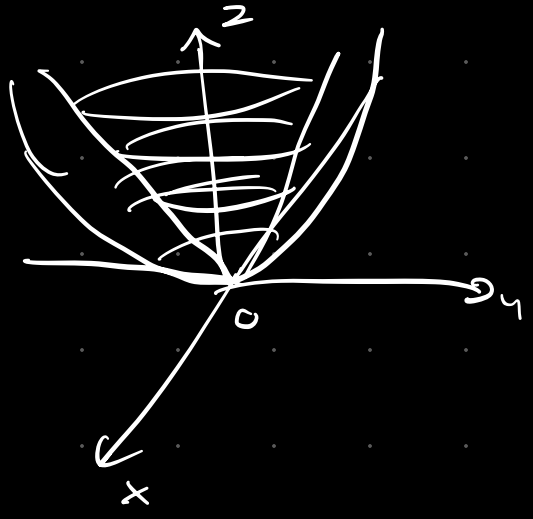
$$= \frac{-16+3+3}{12} = -\frac{10}{12}$$

5) $f(x,y,z)=x^2$

$$2z = x^2+y^2$$

$$z = \frac{x^2+y^2}{2} \quad \text{od}$$

$$z = \sqrt{x^2+y^2} \quad \text{do}$$



cylindrické souřadnice:

$\Delta\Phi = r$ $\quad x = r\cos\varphi$, $\varphi \in \langle 0, 2\pi\rangle$

$y = r\sin\varphi$, $r \in \langle 0, 2\rangle$

$z = z$

$$\frac{x^2+y^2}{2} \le z \le \sqrt{x^2+y^2}$$

$$\frac{r^2}{2} \le z \le r$$

$$\int_0^{2\pi}\int_0^{2}\int_{\frac{r^2}{2}}^{r} r^2\cos^2\varphi \cdot r \, dz\, dr\, d\varphi =$$

$$= \int_0^{2\pi}\int_0^{2} r^3\cos^2\varphi \,[z]_{\frac{r^2}{2}}^{r} \, dr\, d\varphi = \int_0^{2\pi}\int_0^{2} r^3\cos^2\varphi \left(r - \frac{r^2}{2}\right) dr\, d\varphi =$$

$$= \int_0^{2\pi}\int_0^{2} r^4\cos^2\varphi - \frac{r^5}{2}\cos^2\varphi \, dr\, d\varphi = \int_0^{2\pi}\int_0^{2} \cos^2\varphi\left(r^4 - \frac{r^5}{2}\right) dr\, d\varphi$$

$$\int \cos^2\varphi \, d\varphi = \sin\varphi\cos\varphi$$

2. 2. 2020

1) $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$, $D(f) = \mathbb{R}^2 \setminus \{(0,0)\}$

po x $(y=0)$: $\lim\limits_{x\to} \frac{x\cdot 0}{\sqrt{x^2}} = 0$

po y $(x=0)$: $\lim\limits_{y\to 0} \frac{0\cdot y}{\sqrt{y^2}} = 0$

po $y=kx$, $k\neq 0$:

$$\lim_{(x,kx)\to(0,0)} \frac{kx^2}{\sqrt{x^2+k^2x^2}} = \lim_{(x,kx)\to(0,0)} \frac{x^{\cancel{2}}k}{\cancel{x}(\sqrt{1+k^2})} = \lim_{(x,kx)\to(0,0)} x \overbrace{\frac{k}{\sqrt{1+k^2}}}^{c/R} = 0$$

<u>Kandidát</u>: lim je 0, sevření, $\lim_{t\to 0}|f(t)|=0 \Leftrightarrow \lim_{t\to 0} f(t)=0$

$$0 \le \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \le \left|\frac{\left(\sqrt{x^2+y^2}\right)^2}{\sqrt{2\left(\sqrt{x^2+y^2}\right)^2}}\right| = \left|\frac{\left(\sqrt{x^2+y^2}\right)^{\cancel{2}\,1}}{\cancel{\left(\sqrt{x^2+y^2}\right)}\cdot\sqrt{2}}\right| =$$

<u>odhad</u>: $|x|, |y| \le \sqrt{x^2+y^2}$

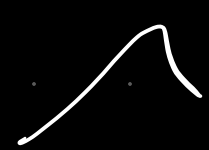
$$= \left|\frac{1}{\sqrt{2}}\sqrt{x^2+y^2}\right| \xrightarrow{(x,y)\to(0,0)} 0$$

z věty o sevření je limita 0.

$$\lim_{(x,y)\to(0,0)} \frac{tg(x)}{y} = \lim_{(x,y)\to(0,0)} \frac{tg(x)}{xy}\cdot x$$

$$\frac{tg(x)}{xy}\cdot x = \underbrace{\frac{\sin(x)}{x}}_{\text{identita}=1} \cdot \underbrace{\frac{1}{\cos(x)}}_{\substack{1\\ \cos(x)\\ \text{v } x=0\\ \text{spojitá}=1}} \cdot \frac{x}{y} \longrightarrow \text{vyšetříme}$$

$\lim_{(x,y)\to(0,0)} \frac{x}{y}$: po ose x: $(y=0)$: $\lim_{x\to 0}\frac{x}{0}$ není definováno / $\pm\infty$

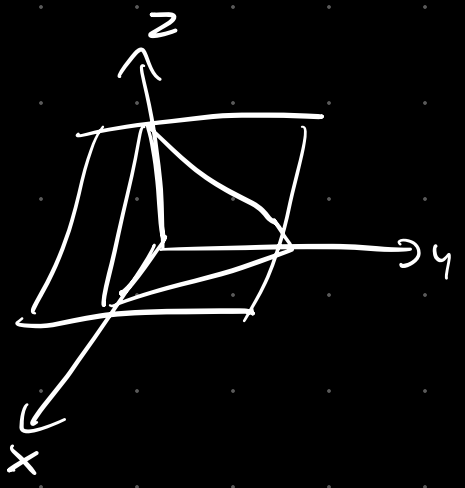


neexistuje / liší se

po ose y $(x=0)$: $\lim_{y\to 0}\frac{0}{y}=0$

limita neexistuje

2)



$z = 6 - 2x - 3y$

$\rho: 2x + 3y + z = 6$

kvádr s hranami $\parallel$ s x, y, z a 1 vrcholom v $(0,0)$

maximálny objem

$V(x,y,z) = x \cdot y \cdot z$

$f(x,y,z) = xyz$, $\nabla f = (yz, xz, xy)$

$g(x,y,z) = 2x + 3y + z - 6$, $\nabla g = (2, 3, 1)$

$yz = 2\lambda \rightarrow \lambda = \frac{yz}{2}$

$xz = 3\lambda \rightarrow \lambda = \frac{xz}{3}$

$xy = \lambda \rightarrow \lambda = xy$

$2x + 3y + z = 6$

+ vylúčim súradnicu, cez ktorú by objem bol 0

$x > 0, y > 0, z > 0$

$\frac{yz}{2} = \frac{xz}{3}$

$\frac{y}{2} = \frac{x}{3}$

$y = \frac{2}{3}x$

$\frac{yz}{2} = xy$

$z = 2x$

$2x + 3\left(\frac{2}{3}x\right) + 2x = 6$

$2x + 2x + 2x = 6$

$6x = 6$

$x = 1$

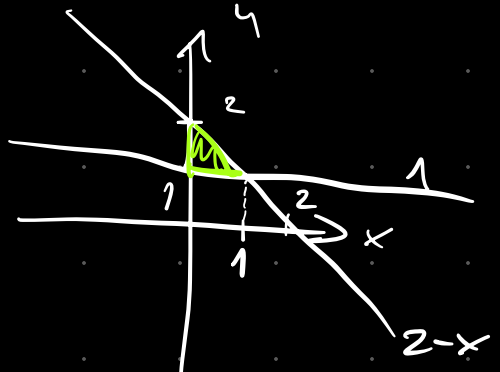
$y = \frac{2}{3}$, $z = 2$

nejvyšší objem s vrcholem v bodě $(1, \frac{2}{3}, 2)$

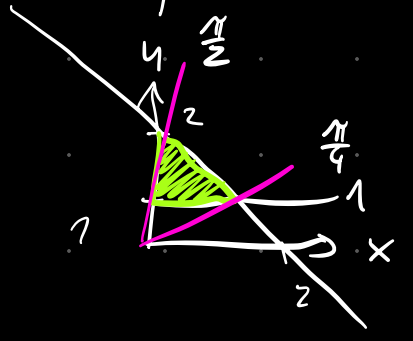
$$V\left(1, \frac{2}{3}, 2\right) = \frac{4}{3}$$

3) $\int_0^1 \int_1^{2-x} f \, dy \, dx$

a) $\int_1^2 \int_0^{2-y} f \, dx \, dy$



b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{1}{\sin\varphi}}^{\frac{2}{\sin\varphi+\cos\varphi}} f(r\cos\varphi, r\sin\varphi) \cdot r \, dr \, d\varphi$



$r\sin\varphi \geq 1$

$r \geq \frac{1}{\sin\varphi}$

$r\sin\varphi \leq 2 - r\cos\varphi$

$r \leq \frac{2}{\sin\varphi + \cos\varphi}$

4) dělka křivky $\vec{r}(t) = (e^{t+1}, e^t \sin t, e^t \cos t)$

$t \in \langle 0, 2 \rangle$

$d\vec{r}(t) = (e^{t+1}, e^t(\sin t - \cos t), e^t(\cos t + \sin t))$

$\|d\vec{r}(t)\| = \sqrt{e^{2t+2} + e^{2t}(\sin t - \cos t)^2 + e^{2t}(\cos t + \sin t)^2} =$

$$= e^t \sqrt{e^2 + \sin^2 t - 2\sin t\cos t + \cos^2 t + \cos^2 t + 2\sin t\cos t + \sin^2 t} =$$

$$d(e^t \sin t) = e^t \sin t - e^t \cos t = e^t(\sin t - \cos t)$$

$$d(e^t \cos t) = e^t \cos t + e^t \sin t =$$

$$= e^t \sqrt{e^2 + 2}$$

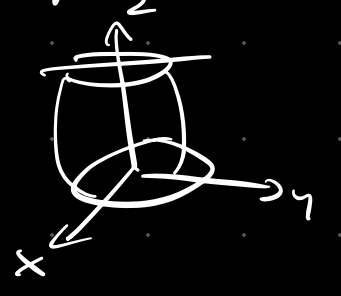
$$\int_0^2 e^t \sqrt{e^2+2}\, dt = \sqrt{e^2+2} \int_0^2 e^t dt =$$

$$= \sqrt{e^2+2}\, [e^t]_0^2 = \sqrt{e^2+2}\,(e^2 - 1)$$

5) Gauss: $\iint_{(M)} \vec{F}\, d\vec{S} = \iiint_M \operatorname{div} \vec{F}\, dV$

$\vec{F} = (3x + y^2,\ y^3 - z,\ -2z^2 + x)$

M je parch dtvara omeđen $x^2 + y^2 = 9,\ z = 0,\ z = 5$



cylindrične koordinate

$x = r\cos\varphi \quad \varphi \in \langle 0, 2\pi \rangle$

$y = r\sin\varphi \quad r \in \langle 0, 3 \rangle$

$z = h \quad h \in \langle 0, 5 \rangle$

$\operatorname{div}\vec{F} = 3 + 3y^2 - 4z \qquad \Delta\Phi = r$

$$\int_0^{2\pi}\int_0^{3}\int_0^{5}\left(3+3(r^3\sin^2\varphi)-4h\right) r\,dh\,dr\,d\varphi =$$

$$= \int_0^{2\pi}\int_0^{3}\int_0^{5}\left(3r+3r^3\sin^2\varphi-4hr\right)dh\,dr\,d\varphi =$$

$$= \int_0^{2\pi}\int_0^{3}\left[3hr+3hr^3\sin^2\varphi-2h^2r\right]_0^5 dr\,d\varphi =$$

$$= \int_0^{2\pi}\int_0^{3} 15r+15r^3\sin^2\varphi-50r\,dr\,d\varphi =$$

$$= \int_0^{2\pi}\int_0^{3} 15r^3\sin^2\varphi-35r\,dr\,d\varphi = \int_0^{2\pi}\left[\frac{15}{4}r^4\sin^2\varphi-\frac{35}{2}r^2\right]_0^3 d\varphi =$$

$$= \int_0^{2\pi}\left(\frac{5}{4}\cdot 3^5\sin^2\varphi-\frac{9\cdot 35}{2}\right)d\varphi = \frac{5}{4}\cdot 3^5\left[\frac{\varphi}{2}-\frac{\sin\varphi\cos\varphi}{2}\right]_0^{2\pi}-\left[\frac{9\cdot 35}{2}\varphi\right]_0^{2\pi}$$

$$= \frac{5}{4}\cdot 3^5(2\pi)-(9\cdot 35\pi) = \frac{5}{2}\cdot 3^5\cdot\pi-315\pi =$$

$$\int \sin^2 \varphi = -\sin\varphi\cos\varphi + \int \underbrace{\cos^2\varphi}_{1-\sin^2\varphi} d\varphi = \frac{\varphi}{2} - \frac{\sin\varphi\cos\varphi}{2} + C$$

$$= \frac{5\cdot 243}{2}\pi - \frac{630}{2}\pi = \frac{1215}{2}\pi - \frac{630}{2}\pi =$$

$$= \frac{585}{2}\pi$$

2020/3

1) $\lim\limits_{(x,y)\to(0,0)} \frac{x\cos y}{\sqrt{x^2+y^2}}$, $D(f) = \mathbb{R}^2 \setminus \{(0,0)\}$

$$= \lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}} \cdot \underbrace{\cos y}_{1} \quad \text{je v } (0,0) \text{ spojité} = 1$$

vyšetřím $\lim\limits_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$, po ose x $(y=0)$

$$\lim_{x\to 0} \frac{x}{\sqrt{x^2}} = \lim_{x\to 0} \frac{x}{|x|} = \pm 1 \quad \lightning$$

nyšou stejné oboustranné limity

neexistuje

$\lim\limits_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^4}$ $\quad D(A) = \mathbb{R}^2 \setminus \{(0,0)\}$

přiblížení po x ($y=0$); po y ($x=0$):

$$\lim_{x\to 0}\frac{0\cdot x}{x^2}=0 \qquad \lim_{y\to 0}\frac{y^q\cdot 0}{y^q}=0$$

po $y=kx$:

$$\lim_{(x,kx)\to(0,0)}\frac{x\,k^q x^q}{\sqrt{x^2+k^2x^2}}=\lim_{(x,kx)\to(0,0)} x^q\,\underbrace{\frac{k^2}{\sqrt{1+k^2}}}_{\in\mathbb{R}}=0$$

odhad v pol. souř.: $x=r\cos\varphi$, $y=r\sin\varphi$, $r\in\mathbb{R}^+$, $\varphi\in\langle 0,2\pi\rangle$

$$\frac{r\cos\varphi\,(r\sin\varphi)^q}{r^q\cos^q\varphi+r^q\sin^q\varphi}=\frac{r^q(\cos\varphi\sin^q\varphi)}{r^q(\cos^q\varphi+\sin^q\varphi)}=r\left(\frac{\cos\varphi\sin^q\varphi}{\cos^q\varphi+\sin^q\varphi}\right)$$

$$0\le\left|r\left(\frac{\cos\varphi\sin^q\varphi}{\cos^q\varphi+\sin^q\varphi}\right)\right|\le|Mr|\xrightarrow{r\to 0}0$$

spojité funkce na $\langle 0,2\pi\rangle$ nabývá svého maxima, označme ho M

z věty o sevření tedy $\lim_{(x,y)\to(0,0)}\frac{xy^q}{x^2+y^q}=0$

2) Kritické body $f(x,y)=x^2+xy^2-16x$

$D(f)=\mathbb{R}^2$ $\quad\nabla f(x,y)=(2x+y^2-16,\,2xy)$

$$H_f(x,y) = \begin{pmatrix} 2 & 2y \\ 2y & 2x \end{pmatrix}$$

$$2x + y^2 - 16 = 0$$

$$2xy = 0 \rightarrow x = 0 \vee y = 0$$

$\underline{y = 0:}$

$$2x - 16 = 0$$
$$x = 8$$

$\underline{x = 0:}$

$$y^2 - 16 = 0$$
$$y = \pm 4$$

kritické body: $(8,0), (0,4), (0,-4)$

1) $(8,0)$

$$H_f(8,0) = \begin{pmatrix} 2 & 0 \\ 0 & 16 \end{pmatrix} \rightarrow \text{pos. def.} \rightarrow \text{lokální maximum}$$

2) $(0,4)$

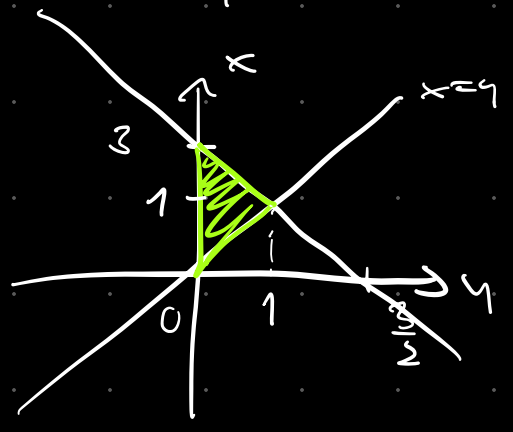
$$H_f(0,4) = \begin{pmatrix} 2 & 8 \\ 8 & 0 \end{pmatrix} \quad \begin{matrix} \Delta_1 > 0 \\ \Delta_2 < 0 \end{matrix} \Big\} \text{indefinitní, sedlo}$$

3) $(0,-4)$

$$H_f(0,-4) = \begin{pmatrix} 2 & -8 \\ -8 & 0 \end{pmatrix} \quad \begin{matrix} \Delta_1 > 0 \\ \Delta_2 < 0 \end{matrix} \Big\} \text{lokální maximum}$$

3)

$$\int_0^1 \int_4^{3-2y} f \, dx \, dy$$



$x = 3 - 2y$

$x - 3 = -2y$

$\frac{3}{2} - \frac{x}{2} = y$

$y = 3 - 2y$

$3y = 3$

$y = 1$

$$\int_0^{\frac{\pi}{4}} \int_0^{\frac{3}{\cos\varphi + 2\sin\varphi}} r \cdot f(r\cos\varphi, r\sin\varphi) \, dr \, d\varphi$$

a)

$$\int_0^1 \int_0^x f \, dy \, dx +$$

$$+ \int_1^3 \int_0^{\frac{3}{2} - \frac{x}{2}} f \, dy \, dx$$

b) $x = r\cos\varphi, \; y = r\sin\varphi, \; \varphi \in \langle 0, \frac{\pi}{4} \rangle$

$x \leq 3 - 2y$

$r\cos\varphi \leq 3 - 2r\sin\varphi$

$r\cos\varphi + 2r\sin\varphi \leq 3$

$r(\cos\varphi + 2\sin\varphi) \leq 3$

$r \leq \frac{3}{\cos\varphi + 2\sin\varphi}$

4) Green: $\int_{(C)} \vec{F}\, d\vec{s} = \iint_M \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dS$

$\vec{F} = (y + e^x, 3x^2 - 5y)$, křivka hladu ori,

$\frac{\partial F_2}{\partial x} = 6x$

$\frac{\partial F_1}{\partial y} = 1$

$x^2 + y^2 = y$

$x^2 + y^2 - y = 0$

$x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4}$

$x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$

kružnice o středu $S(0, \frac{1}{2})$ a $r = \frac{1}{2}$

Polární souřadnice:

$x = r\cos\varphi$

$y = r\sin\varphi + \frac{1}{2}$

$r \in \langle 0, \frac{1}{2} \rangle$

$\varphi \in \langle 0, 2\pi \rangle$

$\Delta\Phi = r$

$$\int_0^{2\pi}\int_0^{\frac{1}{2}} (6r\cos\varphi - 1)\, r\, dr\, d\varphi =$$

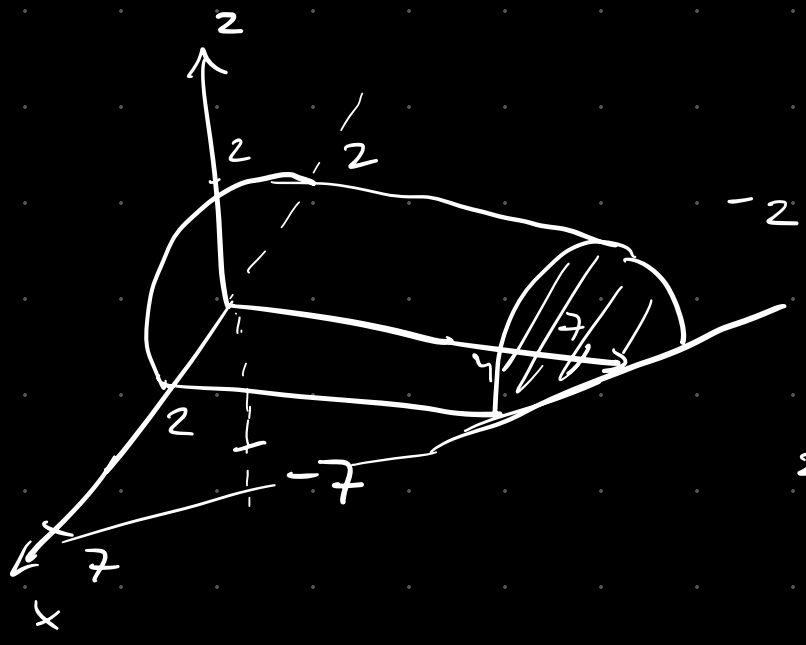
$$= \int_0^{2\pi} \left[2r^3\cos\varphi - \frac{r^2}{2}\right]_0^{\frac{1}{2}} d\varphi =$$

$$= \int_0^{2\pi} \left(\frac{1}{4}\cos\varphi - \frac{1}{8}\right) d\varphi =$$

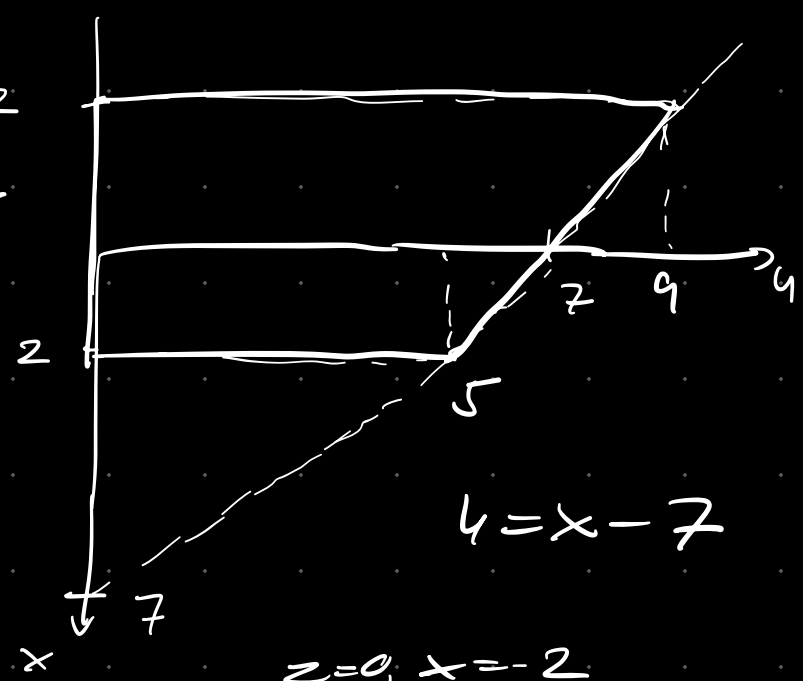
$$= \left[\frac{1}{4}\sin\varphi - \frac{1}{8}\varphi\right]_0^{2\pi} = -\frac{\pi}{4}$$

5) Objem tělesa E daného

$x^2+z^2 \le 4$, $z \ge 0$, $y \ge 0$, $x+y+z \le 7$



$x+y+z-7=0$



$y = x-7$

prohození souřadnic:

Φ_1 $\quad y \to z$, $\quad z \to x$, $\quad x \to y$ $\qquad \Delta\Phi_1 = 1$

$z=0$, $x=-2$

$-2+y-7=0$

$y=9$

$z=0$, $x=2$

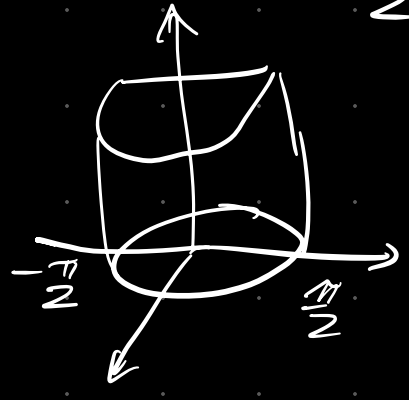
$2+y-7=0$

$y=5$

$x^2+y^2 \le 4$

$z \ge 0$, $x \ge 0$

$x+y+z \le 7$



cylindrické:

$x = r\cos\varphi$ $\qquad \varphi \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$

$y = r\sin\varphi$ $\qquad r \in \langle 0, 2 \rangle$ $\qquad \Delta\Phi = r$

$z = h$

$$z \leq 7 - x - y$$

$$h \leq 7 - r\cos\varphi - r\sin\varphi$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2} \int_{0}^{7 - r(\cos\varphi - \sin\varphi)} r \, dh \, dr \, d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2} 7r - r^2(\cos\varphi - \sin\varphi) \, dr \, d\varphi =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{7}{2} r^2 - \frac{r^3}{3}(\cos\varphi - \sin\varphi) \right]_0^2 d\varphi =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 14 - \frac{8}{3}(\cos\varphi - \sin\varphi) \, d\varphi =$$

$$= \left[14\varphi - \frac{8}{3}\sin\varphi - \frac{8}{3}\cos\varphi \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$= \left(7\pi - \frac{8}{3}\right) - \left(-7\pi + \frac{8}{3}\right) = \underline{14\pi - \frac{16}{3}}$$

2021/9

1) $f(x,y) = e^{x+y}(x^2-2y^2)$

spojitá na $\mathbb{R}^2$, $D(f) = \mathbb{R}^2$

$$\nabla f(x,y) = \left(e^{x+y}(x^2-2y^2-2x),\ e^{x+y}(x^2-2y^2+4y)\right)$$

$$\frac{d}{dx} e^{x+y}(x^2-2y^2) = e^{x+y}(x^2-2y^2) - e^{x+y}2x =$$

$$= e^{x+y}(x^2-2y^2-2x)$$

$$\frac{d}{dy} e^{x+y}(x^2-2y^2) = e^{x+y}(x^2-2y^2) - e^{x+y}(-4y) =$$

$$= e^{x+y}(x^2-2y^2+4y)$$

$$-\begin{cases} x^2-2y^2-2x=0 \\ x^2-2y^2+4y=0 \end{cases}, \quad e^t > 0 \ \forall t \in \mathbb{R}$$

$$-2x-4y=0$$

$$2x=-4y$$

$$x=-2y$$

$$4y^2 - 2y^2 + 4y = 0$$

$$2y^2 + 4y = 0$$

$$2y(y+2) = 0 \rightarrow y_1 = 0 \;,\; y_2 = -2$$

$$\downarrow$$

$$x_1 = 0 \qquad x_2 = 4$$

<u>Kritické body</u>: $(0,0)$, $(-2,4)$

$$H_{f(x,y)} = \begin{pmatrix} & \\ & \end{pmatrix}$$

$$\nabla f(x,y) = \left(e^{x+y}(x^2 - 2y^2 - 2x),\; e^{x+y}(x^2 - 2y^2 + 4y) \right)$$

$$\frac{df}{dx\,dx} = e^{x+y}(x^2 - 2y^2 - 2x) - e^{x+y}(2x - 2) =$$

$$= e^{x+y}(x^2 - 2y^2 - 4x + 2)$$

$$\frac{df}{dx\,dy} = e^{x+y}(x^2 - 2y^2 - 2x) - e^{x+y}(-4y) =$$

$$= e^{x+y}(x^2 - 2y^2 - 2 + 4y)$$

$$\frac{df}{dy\,dy} = e^{x+y}(x^2 - 2y^2 + 4y) - e^{x+y}(-4y + 4) =$$

$= e^{x+y}(x^2 - 2y^2 - y)$

$H_{f(x,y)} = \begin{pmatrix} & \\ & \end{pmatrix}$

2) Extrémy $f(x,y) = xy$, $\nabla f = (y, x)$

$M: g: x^2 + xy + 4y^2 \leq 1$ $\nabla g = (2x + y, x + 8y)$

1) vnitřek – otevřená množina, stačí $\nabla f(x,y) = \vec{0}$

$x^2 + xy + 4y^2 < 1$

$y = 0$
$x = 0$ v $(0,0)$, $(0,0) \in M$

2) hranice: $g(x,y) = x^2 + xy + 4y^2 - 1$

Lagrange: vyloučíme body kde $x = 0, y = 0$, tam je $f = 0$

$y = \lambda(2x + y) \rightarrow \lambda = \frac{y}{2x+y}$

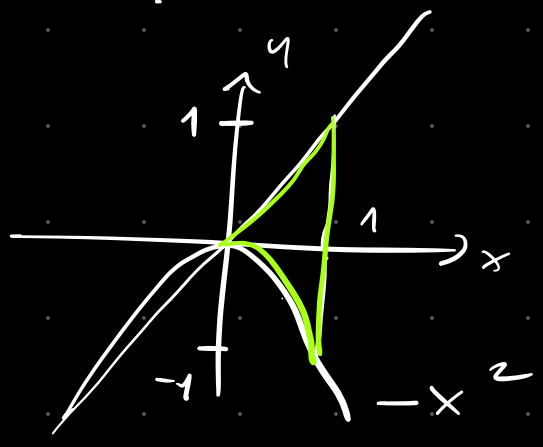
$x = \lambda(x + 8y) \rightarrow \lambda = \frac{x}{x+8y}$

$x^2 + xy + 4y^2 = 1$

$$\frac{y}{2x+y} = \frac{x}{x+8y}$$

$$xy + 8y^2 = 2x^2 + yx$$

3) $\int_0^1 \int_{-x^2}^{x} f \, dy \, dx$



$y = -x^2$

$-y = x^2$

$x = \sqrt{-y}$

$$\int_{-\frac{\pi}{2}}^{0} \int_{0}^{\frac{1}{\cos\varphi}} r \cdot f \, dr \, d\varphi + \int_{0}^{\frac{\pi}{3}} \int_{0}^{\frac{1}{\cos\varphi}} r \cdot f \, dr \, d\varphi$$

$+ g(\varphi)$

$\frac{g(\varphi)}{\cos\varphi}$

a) $\int_{-1}^{0} \int_{\sqrt{-y}}^{1} f \, dx \, dy + \int_{0}^{1} \int_{y}^{1} f \, dx \, dy$

b) $x = r\cos\varphi$

$y = r\sin\varphi$

$\varphi \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$

$-\frac{\pi}{2}$ až 0

$y \geq -x^2$

$r\sin\varphi \geq -r^2\cos^2\varphi$

$\sin\varphi \geq -r\cos^2\varphi$

$\frac{\sin\varphi}{\cos^2\varphi} \leq r$

$\frac{\operatorname{tg}(\varphi)}{\cos\varphi} \leq r$ $\qquad r \leq \frac{1}{\cos\varphi}$

$x \leq 1 \quad r\cos\varphi \leq 1$

4) Green: $\int_{(C)} \vec{F} d\vec{s} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dS$

$\vec{F}(x,y) = (e^x + y, 3x^2 + \cos y)$

$\frac{\partial F_2}{\partial x} = 6x$

$\frac{\partial F_1}{\partial y} = 1$

2021/2

1) $\lim_{(x,y)\to(0,0)} \frac{x^2+y^3}{x^2+y^2}$ $D(f) = \mathbb{R}^2 / \{(0,0)\}$

Přiblížení: Po x $(y=0)$:

$\lim_{x\to 0} \frac{x^2+0^3}{x^2+0^2} = 1$

Po y $(x=0)$:

$\lim_{y\to 0} \frac{0^2+y^3}{0^2+y^2} = 0$ ↯

limita závisí na směru, neexistuje

$\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$

po ose x $(y=0)$:

$\lim_{x\to 0} \frac{(x+0)^2}{x^2+0^2} = 1$

po ose y $(x=0)$

$\lim_{y\to 0} \frac{(0+y)^2}{y^2} = 1$

po $y = kx$:

$\lim_{(x,kx)\to(0,0)} \frac{(x+kx)^2}{x^2+k^2x^2} =$

$$= \frac{(x(1+h))^2}{x^2(1+h^2)} = \frac{x^2(1+h)^2}{x^2(1+h^2)} = \frac{1+2h+h^2}{1+h^2} = 1 + \underbrace{\frac{h^2}{1+h^2}}_{\text{závisí na směru}}$$

2) Elipsa $\frac{x^2}{4} + \frac{y^2}{9} \leq 1$, $=: M$

výška roviny dané funkcí $z = 2x + y$.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| \quad f(x,y)$

$\nabla f(x,y) = (2, 1)$

není nulový pro reálné (x,y), vynecháme tedy vnitřek elipsy (úvaha – výška bude stejně nejvyšší nad krajem)

Lagrange na ∂M

$g(x,y) = \frac{x^2}{4} + \frac{y^2}{9} - 1$, $\nabla g(x,y) = \left(\frac{x}{2}, \frac{2}{9}y\right)$ nulový pouze v $(0,0)$ ale $(0,0) \notin \partial M$

$2 = \lambda \frac{x}{2}$ ← $x, y, \lambda \neq 0$

$1 = \frac{2}{9} y \lambda$

$\lambda = \frac{4}{x}$

$\lambda = \frac{9}{2y}$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

$\frac{4}{x} = \frac{9}{2y}$

$8y = 9x$

$y = \frac{9}{8}x$

$\frac{x^2}{4} + \frac{\frac{81}{64}x^2}{9} = 1$

$$x^2 + \frac{y}{64}x^2 = 1$$

$$\frac{73}{64}x^2 = 1$$

$$x = \pm\frac{8}{\sqrt{73}} \rightarrow y = \pm\frac{9}{\sqrt{73}}$$

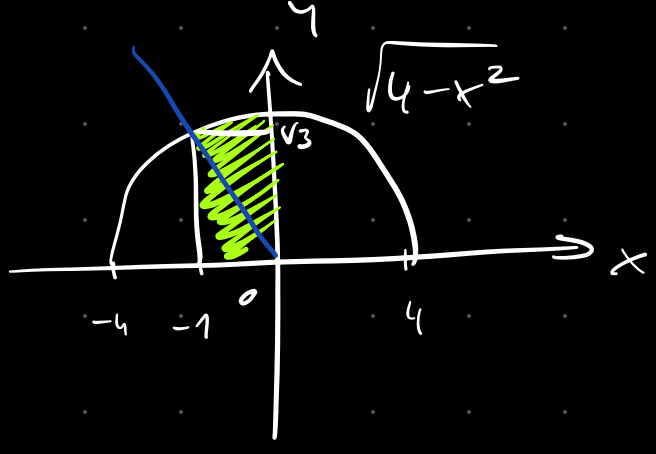
body: $\left(\frac{8}{\sqrt{73}}, \frac{9}{\sqrt{73}}\right), \left(-\frac{8}{\sqrt{73}}, -\frac{9}{\sqrt{73}}\right)$

$$f\left(\frac{8}{\sqrt{73}}, \frac{9}{\sqrt{73}}\right) = \frac{16}{\sqrt{73}} + \frac{9}{\sqrt{73}} = \frac{25}{\sqrt{73}}$$

$$f\left(-\frac{8}{\sqrt{73}}, -\frac{9}{\sqrt{73}}\right) = -\frac{16}{\sqrt{73}} - \frac{9}{\sqrt{73}} = -\frac{25}{\sqrt{73}}$$

Rovina dosahuje nejvíce do $\frac{25}{\sqrt{73}}$ v

bodě $\left(\frac{8}{\sqrt{73}}, \frac{9}{\sqrt{73}}\right)$



3) $$\int_{-1}^{0}\int_{0}^{\sqrt{4-x^2}} f\,dy\,dx$$

$\sqrt{4-(-1)^2} = \sqrt{3}$

a) $$\int_{0}^{\sqrt{3}}\int_{-1}^{0} f\,dx\,dy + \int_{\sqrt{3}}^{4}\int_{-\sqrt{4-y^2}}^{0} f\,dx\,dy$$

$y = \sqrt{9-x^2}$

$y^2 = 9 - x^2$

$x^2 + y^2 = 9$

$x = \sqrt{9-y^2}$

b) polárně!

$x = r\cos\varphi$

$y = r\sin\varphi$

I: $\varphi \in \langle \frac{\pi}{2}, \rangle$ >

$\operatorname{arctg}(-\sqrt{3}) \rightarrow \sin\theta = \frac{\sqrt{3}}{2}$

$\cos\theta = -\frac{1}{\sqrt{2}}$

$\frac{2\pi}{3}$

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \int_{0}^{2} r \cdot f(r\cos\varphi, r\sin\varphi)\, dr\, d\varphi +$$

$$+ \int_{\frac{2\pi}{3}}^{\pi} \int_{0}^{-\frac{1}{\cos\varphi}} r \cdot f(r\cos\varphi, r\sin\varphi)\, dr\, d\varphi$$

II: $x \geq -1$

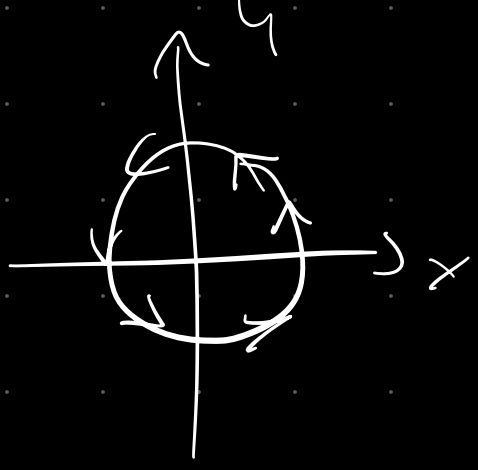
$r\cos\varphi \geq -1$ $\cos\varphi$ zde záporný!

$r \leq -\frac{1}{\cos\varphi}$

4) C: $x^2 + y^2 = 1$ & $x + y + z = 2$

hladce orientovaná při pohledu xy

pole $\vec{F} = (y, z, x)$



Parametrizace:

$\vec{\varphi}(\varphi)$: $x = \cos\varphi$

$y = \sin\varphi$

$z = 2 - \cos\varphi - \sin\varphi$

$\varphi \in \langle 0, 2\pi \rangle$

Hrubý 11. druh

$$d\vec{\varphi}(\varphi) = (-\sin\varphi, \cos\varphi, \sin\varphi - \cos\varphi)$$

$$\int_0^{2\pi} \vec{F}(\cos\varphi, \sin\varphi, 2 - \cos\varphi - \sin\varphi) \cdot (-\sin\varphi, \cos\varphi, \sin\varphi - \cos\varphi) =$$

$$= \int_0^{2\pi} (-\sin^2\varphi + 2\cos\varphi - \cos^2\varphi - \cos\varphi\sin\varphi + \cos\varphi\sin\varphi - \cos^2\varphi)\,d\varphi$$

$$= \int_0^{2\pi} (-1 + 2\cos\varphi - \cos^2\varphi)\,d\varphi = \left[-\varphi + 2\sin\varphi - \frac{\sin(2\varphi)}{2} - \frac{\varphi}{2}\right]_0^{2\pi} =$$

$$= \left(-2\pi + 2\cdot 0 - \frac{0}{2} - \pi\right) = -3\pi$$

$$\int \cos^2\varphi\,d\varphi = \sin\varphi\cos\varphi + \int \underbrace{\sin^2\varphi}_{1-\cos^2\varphi}\,d\varphi = \frac{\sin(2\varphi)}{2} + \frac{\varphi}{2}$$

5) Gauss $\iint_{(\partial M)} \vec{F}\,d\vec{S} = \iiint_M \operatorname{div}\vec{F}\,dV$

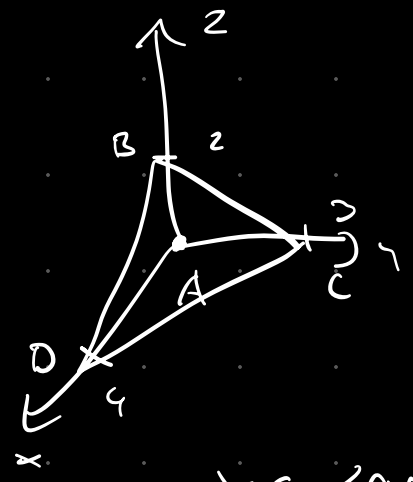
$\vec{F}(x,y,z) = (3xy, y^2z, -x^2y^4)$

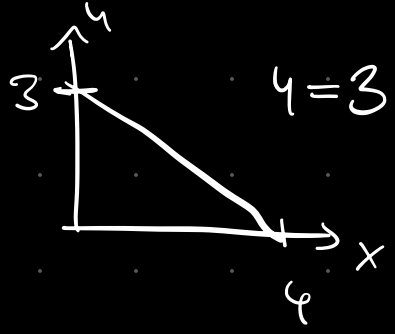
povrch dané kostky: $A=(0,0,0)$

$B=(0,0,2)$

$C=(0,3,0)$

$D=(4,0,0)$





$x \in \langle 0,4 \rangle$

$0 \le y \le 3-\frac{3}{4}x$

$0 \le z \le \frac{12-3x-4y}{6}$

$\operatorname{div}\vec{F} = 3y + 2y + 0$

$= 5y$

2. rovina $\vec{u_1} = (-4,0,2)$

$\vec{u_2} = (-4,3,0)$

$$\begin{vmatrix} -4 & -4 & e_1 \\ 0 & 3 & e_2 \\ 2 & 0 & e_3 \end{vmatrix} = (-6, -8, -12)$$

$-6x - 8y - 12z + d = 0$

$d = 6x + 8y + 12z = 6 \cdot 4 = 24$

$-6x - 8y - 12z + 24 = 0$

$3x + 4y + 6z - 12 = 0$

$z = \frac{12-3x-4y}{6}$

$$\int_0^4 \int_0^{3-\frac{3}{4}x} \int_0^{2-\frac{1}{2}x-\frac{2}{3}y} 5y \, dz\,dy\,dx =$$

$$= \int_0^4 \int_0^{3-\frac{3}{4}x} \left[5yz\right]_0^{2-\frac{1}{2}x-\frac{2}{3}y} dy\,dx = \int_0^4 \int_0^{3-\frac{3}{4}x} \left(10y - \frac{5}{2}xy - \frac{10}{3}y^2\right) dy\,dx$$

$$= \int_0^4 \left[5y^2 - \frac{5}{4}xy^2 - \frac{10}{9}y^3\right]_0^{3-\frac{3}{4}x} dx =$$

$$= \int_0^4 5\left(3-\frac{3}{4}x\right)^2 - \frac{5}{4}x\left(3-\frac{3}{4}x\right)^2 - \frac{10}{9}\left(3-\frac{3}{4}x\right)^3 dx =$$

$$= \int_0^4 45 - \frac{45}{4}x + \frac{45}{16}x^2 - \dots$$

2020/7

1) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

po x $(y=0)$:

$$\lim_{x\to 0} \frac{x^2-0^2}{x^2+0^2} = 1$$

po y $(x=0)$:

$$\lim_{y\to 0} \frac{0^2-y^2}{0^2+y^2} = -1$$

$\Rightarrow$ limita neexistuje

$$\lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^2}$$

po x $(y=0)$:

$$\lim_{x\to 0} \frac{3x\cdot 0^2}{x^2+0^2} = 0$$

po y $(x=0)$:

$$\lim_{y\to 0} \frac{3\cdot 0\cdot y^2}{0^2+y^2} = 0$$

po $y=kx$:

$$\lim_{(x,kx)\to(0,0)} \frac{3x\cdot k^2x^2}{x^2+k^2x^2} = \lim_{(x,kx)\to(0,0)} x\cdot\frac{3k^2}{1+k^2} = 0$$

pozijeu odhad $|x|, |y| \le \sqrt{x^2+y^2}$, věta o sevřené a fakt $\lim_{t\to a} f(a)=0 \Leftrightarrow \lim_{t\to a}(|f(a)|)=0$

$$0 \le \left|\frac{3xy^2}{x^2+y^2}\right| \le_{\text{odhad}} \left|\frac{3(\sqrt{x^2+y^2})^3 \cdot 1}{2(\sqrt{x^2+y^2})^2}\right| = \left|\frac{3}{2}\sqrt{x^2+y^2}\right|$$

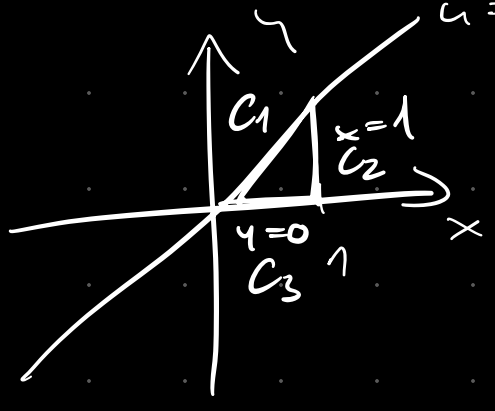
$\left|\frac{3}{2}\sqrt{x^{2}+y^{2}}\right| \xrightarrow{(x,y)\to(0,0)} 0$, tedy z věty o sevření

je původní limita rovna nule

2) $f(x,y) = x^2 - xy + x + y$

na množině

$M: \; 0 \le y \le x, \; x \le 1$



$\nabla f(x,y) = (2x - y + 1, 1 - x)$

<u>uvnitř M</u>

$\begin{pmatrix} 0 < y < 1 \\ 0 < x < 1 \end{pmatrix}$

$2x - y + 1 = 0$

$1 - x = 0$

$x = 1$

$2 - y + 1 = 0$

$-y = -3$

$y = 3$

nezn. extrém v $(1,3)$ ∉ M°

<u>C_1:</u> parametrizace $Q(t) = (t,t)$

$t \in (0,1)$

$f(Q(t)) = t^2 - t^2 + t + t = 2t$

$\frac{df(Q(t))}{dt} = 2 \neq 0$

C_2: parametrizace $\varphi(t) = (1, t)$, $t \in \langle 0,1 \rangle$

$f(\varphi(t)) = 1 - t + 1 + t = 2$

konstantní na C_2

C_3: param. $\varphi(t) = (t, 0)$, $t \in \langle 0,1 \rangle$

$f(\varphi(t)) = t^2 - 0 + t + 0$

$\frac{\partial f(\varphi(t))}{\partial t} = 2t + 1$

$2t + 1 = 0$

$t = -\frac{1}{2} \notin \langle 0,1 \rangle$

v bodech $(0,0)$, $(1,0)$, a $(1,1)$:

$f(0,0) = 0$

$f(1,0) = 1 - 0 + 1 + 0 = 2$

$f(1,1) = 1 - 1 + 1 + 1 = 2$

Funkce je podél úsečky $(1,0)$ až $(1,1)$ konstantní

s funkční hodnotou 2, takže je na ní nejvyšší

1. (20 bodů) Zjistěte (a zdůvodněte) zda následující limity existují a najděte jejich hodnotu:

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}, \qquad \lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^2}.$$

2. (20 bodů) Nalezněte největší a nejmenší hodnotu funkce

$$f(x,y) = x^2 - xy + x + y$$

na množině

$$M: \quad 0 \le y \le x \quad \& \quad x \le 1.$$

3. (20 bodů) Přepište následující integrál

$$\int_1^2 \int_{2-x}^{x} f \, dy \, dx$$

a) v opačném pořadí integrace,

b) v polárních souřadnicích v pořadí $d\varrho\, d\varphi$.

(Polární souřadnice mají střed v počátku.)

4. (20 bodů) Najděte hmotnost drátu ve tvaru kružnice $\mathcal{C}$ popsané rovnicemi

$$x^2 + y^2 + z^2 = 1 \quad \text{a} \quad x = \tfrac{1}{2}$$

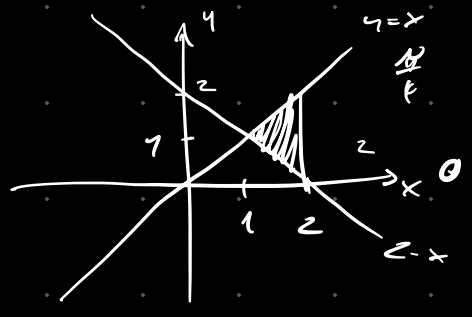
který má (délkovou) hustotu $\delta(x,y,z) = y^2 + |z|$.

5. (20 bodů) Pomocí Gaussovy věty zjistěte, jaký je tok pole

$$\vec{F}(x,y,z) = (z^2 - x,\ -xy + x,\ 3z - y)$$

hranicí tělesa omezeného plochami $z = 4 - y^2$, $z = 0$, $x = 0$ a $x = 3$ s vnější orientací.

3) $\int_1^2 \int_{2-x}^{x} f \, dy \, dx$



a) $\int_0^1 \int_{2-y}^{2} f \, dx \, dy +$

$+ \int_1^2 \int_{y}^{2} f \, dx \, dy$

$\int_0^{\frac{\pi}{4}} \int_{\frac{2}{\sin\varphi+\cos\varphi}}^{\frac{2}{\cos\varphi}} r \cdot f(r\cos\varphi, r\sin\varphi)\, dr\, d\varphi$

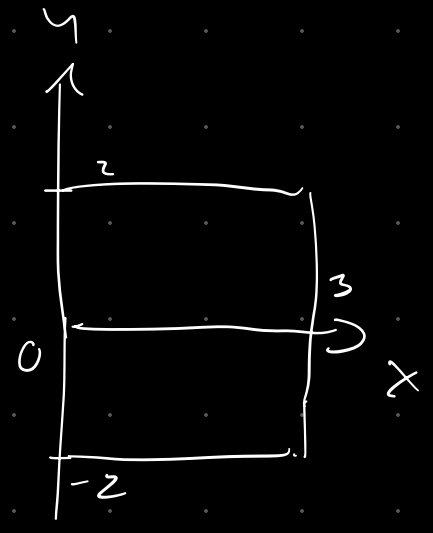
b) $x \le 2 \to r\cos\varphi \le 2 \to r \le \frac{2}{\cos\varphi}$

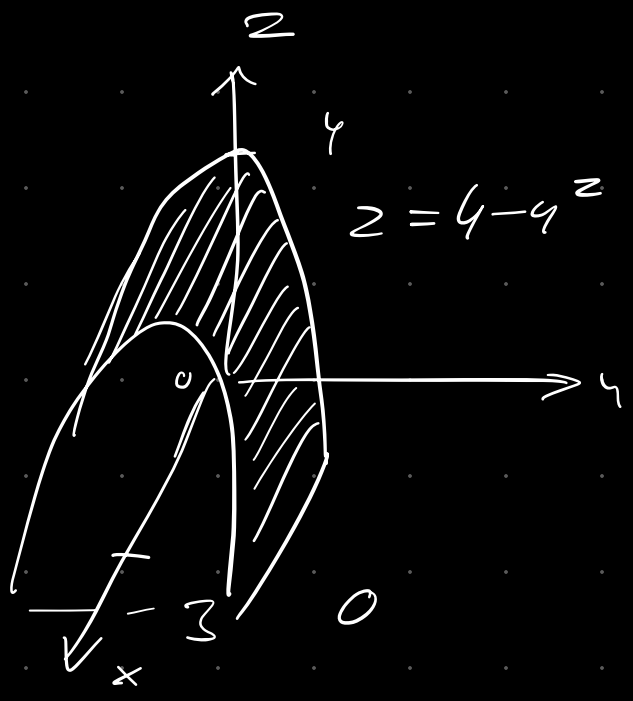
$y \ge 2-x \to r\sin\varphi \ge 2 - r\cos\varphi$

$r \ge \frac{2}{\sin\varphi + \cos\varphi}$

b) Gauss: $\iint\limits_{(M)} \vec{F}\, d\vec{S} = \iiint\limits_{M} div\vec{F}$, M: $z=0$, $x=0$, $x=3$, $z=4-y^2$

$\vec{F} = (z^2-x,\ -xy+x,\ 3z-y)$





$div\vec{F} = -1-x+3 = 2-x$

$$\int_0^3 \int_{-2}^{0} \int_0^{4-y^2} (2-x)\, dz\, dy\, dx = \int_0^3 \int_{-2}^{0} (2-x)[z]_{z=0}^{z=4-y^2}\, dy\, dx =$$

$$= \int_0^3 \int_{-2}^{0} 8-2y^2-4x+4x\, dy\, dx =$$

$$= \int_0^3 \left[8y - \frac{2}{3}y^3 - 4xy + \frac{y^2 x}{3} \right]_{y=-2}^{y=2} dx =$$

$$= \int_0^3 \left(\left(16 - \frac{16}{3} - 8x + \frac{8}{3}x \right) - \left(-16 + \frac{16}{3} + 8x - \frac{8x}{3} \right) \right) dx =$$

$$= \int_0^3 \left(32 - \frac{32}{3} - 16x + \frac{16}{3}x \right) dx$$

$$= \left[32x - \frac{32}{3}x - 8x^2 + \frac{8}{3}x^2 \right]_0^3 =$$

$$= 96 - 32 - 72 + 24 = 120 - 104 = 16$$